Tim and Tom are interesting characters. Their story has been told millions of times and has appeared many times in print. Some people think there is an important moral to their story.

Tim and Tom were twins. They both went to work at age 20 with identical jobs, identical salaries, and at the end of each year, they received identical bonuses of $2000. However, they were not identical in all respects.

Early in life, Tim was conservative and was concerned about his future. Each year he invested his $2000 bonus in a savings program earning 9% interest compounded annually. Tim decided at age 30 to have some fun in life and he began spending his $2000 bonuses on vacations in the Bahamas. This continued until he was 65 years old.

Tom, on the other hand, believed in his youth that life was too short to be concerned about saving for the future. For ten years, he spent his $2000 bonuses on vacations in the Bahamas. At age 30, he began to realize that some day he might not be able to work and then would need funds to provide for his support. He began investing his $2000 bonuses in a savings program earning 9% compounded annually. This continued until he was 65 years old.

Through the years, the brothers became separated. However, they were joyfully reunited at age 65 at a family reunion and exchanged many stories of the events in their lives. Eventually the conversation got around to retirements plans and savings programs. Each brother was proud of his savings and showed the other a spreadsheet describing his savings activities, terms, and accumulations. (See **Table 1** for the results and activities of the two programs.)

The brothers compared their accounts extensively. They were amazed. Tom had made many more $2000 deposits than Tim. Yet, Tim had over $200,000 more than Tom. Tom was perplexed. He exclaimed, "How could there be such a large difference?" They even discussed which plan was best.

Which of these two plans do you think is best? Which would you prefer to follow in a life-long savings program? Before we choose one of these programs, we should make sure the figures in **Table 1** are correct and that we understand the mathematics. It would be unfortunate to base the choice of a life-long savings program on figures which were incorrect, or even on figures which were not completely understood. We should answer Tom's question, "How could there be such a large difference?"

Let us examine very carefully several entries in Tom's part of the table. Most students could easily justify the $2180.00 balance at the end of Year 11 by calculating

$$2000 + 0.09(2000) = 2000(1 + 0.09) = \$2180.00.$$

For the balance at the end of Year 12, they would calculate

$$[2000(1.09) + 2000](1.09) = 2000(1.09)^2 + 2000(1.09) = \$4556.20.$$

Table 1

The Story of Tim and Tom at 9%

Year	Deposits By Tim	Annual Interest Rate	Balance	Deposits By Tom	Balance
1	$2,000.00	9.00%	$2,180.00	$0.00	$0.00
2	$2,000.00	9.00%	$4,556.20	$0.00	$0.00
3	$2,000.00	9.00%	$7,146.26	$0.00	$0.00
4	$2,000.00	9.00%	$9,969.42	$0.00	$0.00
5	$2,000.00	9.00%	$13,046.67	$0.00	$0.00
6	$2,000.00	9.00%	$16,400.87	$0.00	$0.00
7	$2,000.00	9.00%	$20,056.95	$0.00	$0.00
8	$2,000.00	9.00%	$24,042.07	$0.00	$0.00
9	$2,000.00	9.00%	$28,385.86	$0.00	$0.00
10	$2,000.00	9.00%	$33,120.59	$0.00	$0.00
11	$0.00	9.00%	$36,101.44	$2,000.00	$2,180.00
12	$0.00	9.00%	$39,350.57	$2,000.00	$4,556.20
13	$0.00	9.00%	$42,892.12	$2,000.00	$7,146.26
14	$0.00	9.00%	$46,752.41	$2,000.00	$9,969.42
15	$0.00	9.00%	$50,960.13	$2,000.00	$13,046.67
16	$0.00	9.00%	$55,546.54	$2,000.00	$16,400.87
17	$0.00	9.00%	$60,545.73	$2,000.00	$20,056.95
18	$0.00	9.00%	$65,994.84	$2,000.00	$24,042.07
19	$0.00	9.00%	$71,934.38	$2,000.00	$28,385.86
20	$0.00	9.00%	$78,408.47	$2,000.00	$33,120.59
21	$0.00	9.00%	$85,465.24	$2,000.00	$39,281.44
22	$0.00	9.00%	$93,157.11	$2,000.00	$43,906.77
23	$0.00	9.00%	$101,541.25	$2,000.00	$50,038.38
24	$0.00	9.00%	$110,679.96	$2,000.00	$56,721.83
25	$0.00	9.00%	$120,641.16	$2,000.00	$64,006.80
26	$0.00	9.00%	$131,498.86	$2,000.00	$71,947.41
27	$0.00	9.00%	$143,333.76	$2,000.00	$80,602.68
28	$0.00	9.00%	$156,233.80	$2,000.00	$90,036.92
29	$0.00	9.00%	$170,294.84	$2,000.00	$100,320.24
30	$0.00	9.00%	$185,621.37	$2,000.00	$111,529.06
31	$0.00	9.00%	$202,327.30	$2,000.00	$123,746.68
32	$0.00	9.00%	$220,536.75	$2,000.00	$137,063.88
33	$0.00	9.00%	$240,385.06	$2,000.00	$151,579.63
34	$0.00	9.00%	$262,019.72	$2,000.00	$167,401.79
35	$0.00	9.00%	$285,601.49	$2,000.00	$184,647.95
36	$0.00	9.00%	$311,305.63	$2,000.00	$203,446.27
37	$0.00	9.00%	$339,323.13	$2,000.00	$223,936.43
38	$0.00	9.00%	$369,862.21	$2,000.00	$246,270.71
39	$0.00	9.00%	$403,149.81	$2,000.00	$270,615.08
40	$0.00	9.00%	$439,433.30	$2,000.00	$297,150.43
41	$0.00	9.00%	$478,982.28	$2,000.00	$326,073.97
42	$0.00	9.00%	$522,090.70	$2,000.00	$357,600.63
43	$0.00	9.00%	$569,078.86	$2,000.00	$391,964.69
44	$0.00	9.00%	$620,295.96	$2,000.00	$429,421.51
45	$0.00	9.00%	$676,122.60	$2,000.00	$470,249.45

So the figures for Years 11 and 12 are correct and we understand them.

Although our calculations are encouraging, let us use them to build a table and to look for a pattern so as to avoid having to do 35 separate rows of calculations. (See **Table 2**.)

Table 2

Derivation of the Sum of an Annuity Due

End of Year	Balance
11	$2000(1.09)$
12	$(2000(1.09) + 2000)(1.09)$ $= 2000(1.09)^2 + 2000(1.09)$
13	$(2000(1.09)^2 + 2000(1.09)$ $+ 2000)(1.09) = 2000(1.09)^3$ $+ 2000(1.09)^2 + 2000(1.09)$

Extending this pattern, the balance for the end of Year 45 would be

$$2000(1.09)^{35} + 2000(1.09)^{34} + \ldots + 2000(1.09)^2 + 2000(1.09).$$

This sum should be the $470,249.45 given in Table 1 for Tom's balance at the end of Year 45. Let us call this sum S and write equations for S and $(1.09)S$.

$$(1.09)S = 2000(1.09)^{36} + 2000(1.09)^{35} + \ldots + 2000(1.09)^3 + 2000(1.09)^2$$
$$S = 2000(1.09)^{35} + \ldots + 2000(1.09)^3 + 2000(1.09)^2 + 2000(1.09).$$

Next we subtract S from $(1.09)S$ on both sides of the equations to have $(1.09)S - S = 2000(1.09)^{36} - 2000(1.09)$. This simplification occurs since the middle terms subtract out. For example, the $2000(1.09)^{35}$ is found in both S and in $(1.09)S$, and thus they subtract to zero. Continuing to simplify, we have $0.09S = 2000[(1.09)^{36} - 1.09]$ so that

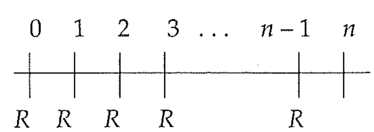
$$S = 2000\left[\frac{(1.09)^{35} - 1}{0.09}\right](1.09).$$

Do this calculation on your scientific calculator to see if you get the expected $470,249.45 . A typical code would be:

$1.09y^x\ 35 - 1 = \div\ 0.09 \times 1.09 \times 2000 =$ and read 470,249.44 .

This result should be quite gratifying since it saved us much work at calculating, and saved several possible errors and an error in the final answer. In addition, we can use this formula for other calculations.

Since we expect to use this more efficient formula for other calculations, we should make it into a general formula. Let us consider the following time-line.



Here, let R denote each of n regular deposits at the beginning of each of n years with deposits earning interest at the rate i. For this, a table similar to Table 2 will be built. (See **Table 3**.)

Table 3

Derivation of the Formula for the Sum of an Annuity Due

End of Year	Balance
1	$R + iR = R(1 + i)$
2	$(R(1 + i) + R)(1 + i) = R(1 + i)^2 + R(1 + i)$
3	$(R(1+i)^2 + R(1 + i) + R)(1 + i)$ $= R(1 + i)^3 + R(1 + i)^2 + R(1 + i)$
.	
.	Extending this pattern gives
.	
n	$S = R(1 + i)^n + R(1 + i)^{n-1} + \ldots + R(1 + i)$

Let us write equations for S and $(1 + i)S$ and subtract S from $(1 + i)S$:

$$(1 + i)S = R(1 + i)^{n+1} + R(1 + i)^n + \ldots + R(1 + i)^2.$$

$$S = R(1 + i)^n + \ldots + R(1 + i)^2 + R(1 + i).$$

$$(1 + i)S - S = R(1 + i)^{n+1} - R(1 + i).$$

$$iS = R[(1 + i)^{n+1} - (1 + i)].$$

Table 4

Years 11 to 45 for Tim

End of Year	Balance
11	33,120.59 + (0.09)(33,120.59) = 33,120.59(1 + 0.09) = 36,101.44
12	33,120.59(1.09) + 0.09(33,120.59)(1.09) = $33{,}120.59(1.09)^2$ = 39,350.57
13	$33{,}120.59(1.09)^2$ + $0.09(33{,}120.59)(1.09)^2$ = $33{,}120.59(1.09)^3$ = 42,892.12
. . .	Extending this pattern gives
45	$(33{,}120.59)(1.09)^{35}$ = 676,122.66

$$A = P \cdot \left[\frac{(1+r)^t - 1}{r} \right] \cdot (1+r)$$

A – End value

P – Amount of each regular deposit

r – Interest rate as a decimal

t – The number of investing periods

So far, we have examined carefully Table 1 for all except Years 11 to 45 for Tim. At the beginning of Year 11, Tim has $33,120.59 (the same as at the end of Year 10). Let us build a table to check the remaining balances. (See **Table 4.**)

By doing these algebraic manipulations and using our calculator we have verified Tim's balance at the end of Year 45. You will notice that our calculator results are a few cents different from those in Table 1 which were obtained by a spreadsheet program on a computer. Check the figures in Table 4 with your calculator.

These calculations suggest the possibility of deriving a general formula which could be used in such cases. Let us label the initial $33,120.59 as *P* and call it *principal*. We will label the interest rate *i* and build a table. (See **Table 5.**)

This derivation gives the *Compound Interest Formula*:

$$A = P \cdot (1+r)^t$$

A – End amount, P – Deposit at the start, t – # of years letting the money ride

Table 5

Derivation of the Compound Interest Formula

End of Year	Balance
0	P
1	$P + iP = P(1+i)$
2	$P(1+i) + iP(1+i) = P(1+i)^2$
3	$P(1+i)^2 + iP(1+i)^2 = P(1+i)^3$
. . .	Extending this pattern gives
n	$S = P(1+i)^{n-1} + iP(1+i)^{n-1} = P(1+i)^n$

Let us summarize what has been done so far and then return to the discussion of the merits of Tim and Tom's investment styles.

1) We have used algebra and our calculator to verify the correctness of the figures in Table 1.
2) We have used algebra to personally understand the mathematics in Table 1.
3) The general Formula for the Sum of an Annuity Due has been derived.
4) The Compound Interest Formula has been derived.
5) We have answered Tom's question: "How could there be such a large difference?"

To further investigate the activities of Tim and Tom, and to judge the merits of their investment styles, do the following You Try Its. ❑

You Try It #1

At age 65 (at the end of Year 45), (a) How much money had Tim accumulated? (b) How much had Tom accumulated? (c) How much more did Tim have than Tom? (d) How much money had Tim contributed to his savings program? (e) How much money had Tom contributed? (f) How many vacations did Tim take in the Bahamas? (g) How many vacations did Tom take in the Bahamas? (h) For how many years did Tim save $2000 and at what ages? (i) For how many years did Tom save $2000 and at what ages?

You Try It #2

(a) Use the *Formula for the Sum of an Annuity Due* to to verify the sum of $33,120.59 for Tim's balance at the end of Year 10. Round final answers to the nearest cent. (b) Write a calculator code which does all the calculations without reentering intermediate results by hand.

You Try It #3

Carefully write out a derivation similar to that in Table 2 to obtain Tim's balance of $33,120.59 at the end of Year 10. If you ever forget the Formula for the Sum of an Annuity Due, you can use this method.

You Try It #4

Use the Compound Interest Formula to verify Tim's balance at the end of (a) Year 35, (b) Year 45.

You Try It #5

Examine the relative merits of Tim and Tom's investment programs by doing the following: (a) Write four or more advantages of Tim's program. (b) Write four or more advantages of Tom's program. (c) Write an argument for the superiority of one of the programs.
(d) Conduct a poll of your class to determine which program they prefer.

You Try It #6

Financial experts think there is a moral to the story of Tim and Tom. What is it? Write your response.

You Try It #7

(a) How much would Tim have if he had invested his $2000 bonus for all 45 years? (b) How much would a person need to invest for 45 years at 9% to reach a retirement goal of two million dollars?

You Try It #8

Is it possible that at a different interest rate, Tom would catch up to Tim by Year 45? Investigate this question by any method you wish. Use a spreadsheet, calculator, computer program, graphing package, mathematics, or any other resource. Write a description of the question, your methods, and results.

Some Answers to the You Try Its

2

b A typical calculator code: 1.09 y^x 10 – 1 = ÷ 0.09 x 1.09 x 2000 = and read 33,120.59

3

End of Year	Balance
1	$2000 + 0.09(2000) = 2000(1.09) = 2180.00$
2	$[2000(1.09) + 2000](1.09) = 2000(1.09)^2 + 2000(1.09) = 4556.20$
3	$[2000(1.09)^2 + 2000(1.09) + 2000](1.09) = 2000(1.09)^3 + 2000(1.09)^2 + 2000(1.09)$
.	
.	
.	
10	$S = 2000(1.09)^{10} + 2000(1.09)^9 + \ldots + 2000(1.09)^2 + 2000(1.09)$

$$1.09S = 2000(1.09)^{11} + 2000(1.09)^{10} + \ldots + 2000(1.09)^3 + 2000(1.09)^2$$

$$1.09S - S = 2000(1.09)^{11} - 2000(1.09)$$

$$0.09S = 2000[(1.09)^{10} - 1](1.09)$$

$$S = 2000\left[\frac{(1.09)^{10} - 1}{0.09}\right](1.09)$$

Typical calculator code:
1.09 y^x 10 – 1 = ÷ 0.09 x 1.09 x 2000 = and read 33,120.59.

This is Tim's balance at the end of Year 10.

4

a $S = 33{,}120.59(1 + 0.09)^{25} = 285{,}601.52$ is Tim's balance at the end of Year 35.

5

a Four advantages of Tim's program:

- Tim had more money at age 65.
- Tim contributed only $20,000 while Tom contributed $70,000.
- Tim took 35 vacations in the Bahamas while Tom took only 10.
- Tim had fun while he was young doing things at home such as working in the yard or playing baseball during his vacations.

7

a $S = \$1{,}146{,}372.$

b $R = \$3{,}489.27.$