LESSON 2-1

Power and Radical Functions

Then

- You analyzed parent functions and their families of graphs. (Lesson 1-5)

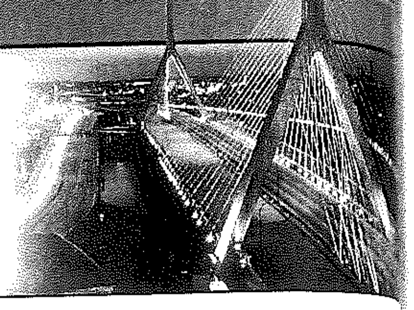
Now

- 1 Graph and analyze power functions.

  2 Graph and analyze radical functions, and solve radical equations.

Why?

- Suspension bridges are used to span long distances by hanging, or suspending, the main deck using steel cables. The amount of weight that a steel cable can support is a function of the cable's diameter and can be modeled by a power function.



NewVocabulary

power function
monomial function
radical function
extraneous solution

1 Power Functions

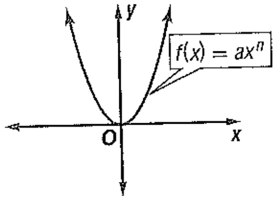
In Lesson 1-5, you studied several parent functions that can be classified as power functions. A **power function** is any function of the form $f(x) = ax^n$, where a and n are nonzero constant real numbers.

A power function is also a type of monomial function. A **monomial function** is any function that can be written as $f(x) = a$ or $f(x) = ax^n$, where a and n are nonzero constant real numbers.

KeyConcept Monomial Functions

Let f be the power function $f(x) = ax^n$, where n is a positive integer.

n Even, a Positive



Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

x- and y-Intercept: 0

Continuity: continuous for $x \in \mathbb{R}$

Symmetry: y-axis

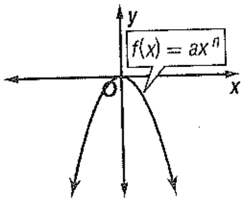
Minimum: $(0, 0)$

Decreasing: $(-\infty, 0)$

Increasing: $(0, \infty)$

End behavior: $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$

n Even, a Negative



Domain: $(-\infty, \infty)$

Range: $(-\infty, 0]$

x- and y-Intercept: 0

Continuity: continuous for $x \in \mathbb{R}$

Symmetry: y-axis

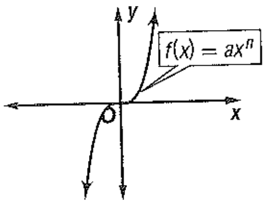
Maximum: $(0, 0)$

Decreasing: $(0, \infty)$

Increasing: $(-\infty, 0)$

End behavior: $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$

n Odd, a Positive



Domain and Range: $(-\infty, \infty)$

x- and y-Intercept: 0

Continuity: continuous on $(-\infty, \infty)$

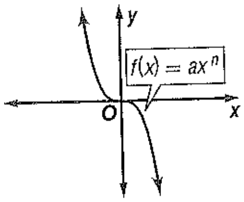
Symmetry: origin

Extrema: none

Increasing: $(-\infty, \infty)$

End Behavior: $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$

n Odd, a Negative



Domain and Range: $(-\infty, \infty)$

x- and y-Intercept: 0

Continuity: continuous for $x \in \mathbb{R}$

Symmetry: origin

Extrema: none

Decreasing: $(-\infty, \infty)$

End Behavior: $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$

ReviewVocabulary

Degree of a Monomial The sum of the exponents of the variables of a monomial.

Monomial functions with an even degree are also *even* in the sense that $f(-x) = f(x)$. Likewise, monomial functions with an odd degree are also *odd*, or $f(-x) = -f(x)$.

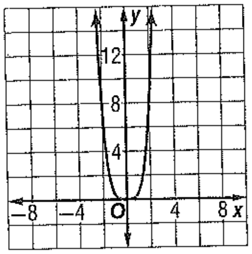
Example 1 Analyze Monomial Functions

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

a. $f(x) = \frac{1}{2}x^4$

Evaluate the function for several x-values in its domain. Then use a smooth curve to connect each of these points to complete the graph.

x	−3	−2	−1	0	1	2	3
$f(x)$	40.5	8	0.5	0	0.5	8	40.5



Domain: $(-\infty, \infty)$ Range: $[0, \infty)$

Intercept: 0

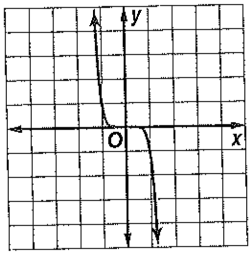
End behavior: $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$

Continuity: continuous on $(-\infty, \infty)$

Decreasing: $(-\infty, 0)$ Increasing: $(0, \infty)$

b. $f(x) = -x^7$

x	−3	−2	−1	0	1	2	3
$f(x)$	2187	128	1	0	−1	−128	−2187



Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

Intercept: 0

End behavior: $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$

Continuity: continuous on $(-\infty, \infty)$

Decreasing: $(-\infty, \infty)$

GuidedPractice

1A. $f(x) = 3x^6$

1B. $f(x) = -\frac{2}{3}x^5$

ReviewVocabulary

Reciprocal Functions Reciprocal functions have the form $f(x) = \frac{a}{x}$. (Lesson 1-5)

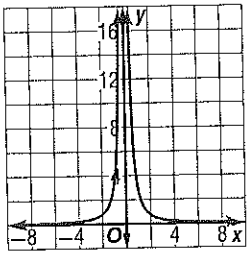
Recall that $f(x) = \frac{1}{x}$ or x^{-1} is undefined at $x = 0$. Similarly, $f(x) = x^{-2}$ and $f(x) = x^{-3}$ are undefined at $x = 0$. Because power functions can be undefined when $n < 0$, the graphs of these functions will contain discontinuities.

Example 2 Functions with Negative Exponents

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

a. $f(x) = 3x^{-2}$

x	−3	−2	−1	0	1	2	3
$f(x)$	$0.\overline{3}$	0.75	3	undefined	3	0.75	$0.\overline{3}$



Domain: $(-\infty, 0) \cup (0, \infty)$ Range: $(0, \infty)$

Intercepts: none

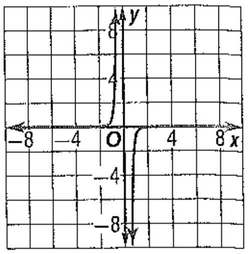
End behavior: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$

Continuity: infinite discontinuity at $x = 0$

Increasing: $(-\infty, 0)$ Decreasing: $(0, \infty)$

b. $f(x) = -\frac{3}{4}x^{-5}$

x	−3	−2	−1	0	1	2	3
f(x)	0.0031	0.0234	0.75	undefined	−0.75	−0.0234	−0.0031



Domain: $(-\infty, 0) \cup (0, \infty)$ Range: $(-\infty, 0) \cup (0, \infty)$

Intercepts: none

End behavior: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$

Continuity: infinite discontinuity at $x = 0$

Increasing: $(-\infty, 0)$ and $(0, \infty)$

GuidedPractice

2A. $f(x) = -\frac{1}{2}x^{-4}$ **2B.** $f(x) = 4x^{-3}$

ReviewVocabulary

Rational Exponents exponents written as fractions in simplest form. (Lesson 0-4)

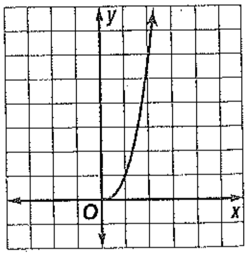
Recall that $x^{\frac{1}{n}}$ indicates the nth root of x, and $x^{\frac{p}{n}}$, where $\frac{p}{n}$ is in simplest form, indicates the nth root of x^p. If n is an even integer, then the domain must be restricted to nonnegative values.

Example 3 Rational Exponents

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

a. $f(x) = x^{\frac{5}{2}}$

x	0	1	2	3	4	5	6
f(x)	0	1	5.657	15.588	32	55.902	88.182



Domain: $[0, \infty)$ Range: $[0, \infty)$

x- and y-Intercepts: 0

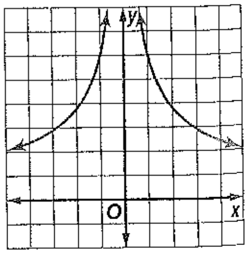
End behavior: $\lim_{x \to \infty} f(x) = \infty$

Continuity: continuous on $[0, \infty)$

Increasing: $(0, \infty)$

b. $f(x) = 6x^{-\frac{2}{3}}$

x	−3	−2	−1	0	1	2	3
f(x)	2.884	3.780	6	undefined	6	3.780	2.884



Domain: $(-\infty, 0) \cup (0, \infty)$ Range: $(0, \infty)$

Intercepts: none

End behavior: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$

Continuity: infinite discontinuity at $x = 0$

Increasing: $(-\infty, 0)$ Decreasing: $(0, \infty)$

GuidedPractice

3A. $f(x) = 2x^{\frac{3}{4}}$ **3B.** $f(x) = 10x^{\frac{5}{3}}$

Real-WorldLink

A Calorie is a unit of energy equal to the amount of heat needed to raise the temperature of one kilogram of water by 1°C. One Calorie is equivalent to 4.1868 kilojoules. The average apple contains 60 Calories.

Source: *Foods & Nutrition Encyclopedia*

StudyTip

Regression Model A polynomial function with rounded coefficients will produce estimates different from values calculated using the unrounded regression equation. From this point forward, you can assume that when asked to use a model to estimate a value, you are to use the unrounded regression equation.

Example 4 Power Regression

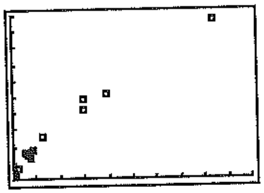
BIOLOGICAL SCIENCE **The following data represents the resting metabolic rate R in kilocalories per day for the mass m in kilograms of several selected animals.**

m	0.3	0.4	0.7	0.8	0.85	2.4	2.6	5.5	6.4	6
R	28	35	54	66	46	135	143	331	293	292
m	7	7.9	8.41	8.5	13	29.3	29.8	39.5	83.6	
R	265	327	346	363	520	956	839	1036	1948	

Source: *American Journal of Physical Anthropology*

a. Create a scatter plot of the data.

The scatter plot appears to resemble the square root function, which is a power function. Therefore, test a power regression model.



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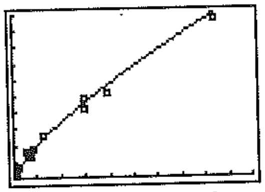
b. Write a polynomial function to model the data set. Round each coefficient to the nearest thousandth, and state the correlation coefficient.

Using the PwrReg tool on a graphing calculator and rounding each coefficient to the nearest thousandth yields $f(x) = 69.582x^{0.759}$. The correlation coefficient r for the data, 0.995, suggests that a power regression may accurately reflect the data.

We can graph the complete (unrounded) regression by sending it to the Y= menu. In the Y= menu, pick up this regression equation by entering VARS, Statistics, EQ. Graph this function and the scatter plot in the same viewing window. The function appears to fit the data reasonably well.



[0, 100] scl: 10 by [0, 2000] scl: 200

c. Use the equation to predict the resting metabolic rate for a 60-kilogram animal.

Use the CALC feature on the calculator to find $f(60)$. The value of $f(60)$ is about 1554, so the resting metabolic rate for a 60-kilogram animal is about 1554 kilocalories.

GuidedPractice

4. CARS The table shows the braking distance in feet at several speeds in miles per hour for a specific car on a dry, well-paved roadway.

Speed	10	20	30	40	50	60	70
Distance	4.2	16.7	37.6	66.9	104.5	150.5	204.9

A. Create a scatter plot of the data.

B. Determine a power function to model the data.

C. Predict the braking distance of a car going 80 miles per hour.

2 Radical Functions

An expression with rational exponents can be written in radical form.

Exponential Form		**Radical Form**
$x^{\frac{p}{n}}$	=	$\sqrt[n]{x^p}$

Power functions with rational exponents represent the most basic of radical functions. A **radical function** is a function that can be written as $f(x) = \sqrt[n]{x^p}$, where n and p are positive integers greater than 1 that have no common factors. Some examples of radical functions are shown below.

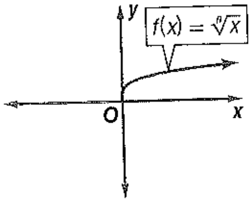
$$f(x) = 3\sqrt{5x^3} \qquad f(x) = -5\sqrt[3]{x^4 + 3x^2 - 1} \qquad f(x) = \sqrt[4]{x + 12} + \frac{1}{2}x - 7$$

It is important to understand the characteristics of the graphs of radical functions as well.

KeyConcept Radical Functions

Let f be the radical function $f(x) = \sqrt[n]{x}$ where n is a positive integer.

n Even



Domain and Range: $[0, \infty)$

x- and y-Intercept: 0

Continuity: continuous on $[0, \infty)$

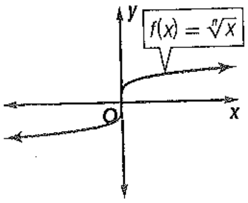
Symmetry: none

Increasing: $(0, \infty)$

Extrema: absolute minimum at $(0, 0)$

End Behavior: $\lim_{x \to \infty} f(x) = \infty$

n Odd



Domain and Range: $(-\infty, \infty)$

x- and y-Intercept: 0

Continuity: continuous on $(-\infty, \infty)$

Symmetry: origin

Increasing: $(-\infty, \infty)$

Extrema: none

End Behavior: $\lim_{x \to -\infty} f(x) = -\infty$ and

$\lim_{x \to \infty} f(x) = \infty$

Example 5 Graph Radical Functions

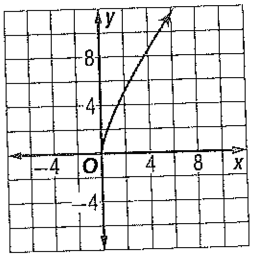
Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

WatchOut!

Radical Functions Remember that when n is even, the domain and range will have restrictions.

a. $f(x) = 2\sqrt[4]{5x^3}$

x	0	1	2	3	4	5
f(x)	0	2.99	5.03	6.82	8.46	10



Domain and Range: $[0, \infty)$

x- and y-Intercepts: 0

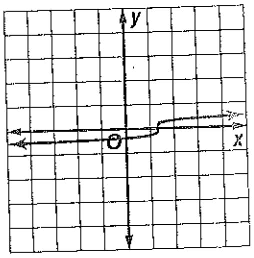
End behavior: $\lim_{x \to \infty} f(x) = \infty$

Continuity: continuous on $[0, \infty)$

Increasing: $(0, \infty)$

b. $f(x) = \frac{1}{4}\sqrt[5]{6x - 8}$

x	−3	−2	−1	0	1	2	3
f(x)	−0.48	−0.46	−0.42	−0.38	−0.29	0.33	0.40



Domain and Range: $(-\infty, \infty)$

x-Intercept: $\frac{4}{3}$

y-Intercept: about −0.38

End behavior: $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$

Continuity: continuous on $(-\infty, \infty)$

Increasing: $(-\infty, \infty)$

GuidedPractice

5A. $f(x) = -\sqrt[3]{12x^2 - 5}$

5B. $f(x) = \frac{1}{2}\sqrt[4]{2x^3 - 16}$

Like radical functions, a radical equation is any equation in which a variable is in the radicand. To solve a radical equation, first isolate the radical expression. Then raise each side of the equation to a power equal to the index of the radical to eliminate the radical.

Raising each side of an equation to a power sometimes produces **extraneous solutions**, or solutions that do not satisfy the original equation. It is important to check for extraneous solutions.

Example 6 Solve Radical Equations

Solve each equation.

a. $2x = \sqrt{100 - 12x} - 2$

$$\begin{aligned}
2x &= \sqrt{100 - 12x} - 2 && \text{Original equation}\\
2x + 2 &= \sqrt{100 - 12x} && \text{Isolate the radical.}\\
4x^2 + 8x + 4 &= 100 - 12x && \text{Square each side to eliminate the radical.}\\
4x^2 + 20x - 96 &= 0 && \text{Subtract } 100 - 12x \text{ from each side.}\\
4(x^2 + 5x - 24) &= 0 && \text{Factor.}\\
4(x + 8)(x - 3) &= 0 && \text{Factor.}\\
x + 8 = 0 \quad &\text{or} \quad x - 3 = 0 && \text{Zero Product Property}\\
x = -8 \quad & \quad\quad\; x = 3 && \text{Solve.}
\end{aligned}$$

Study Tip

Common Factors Remember that you can sometimes factor out a common multiple before using any other factoring methods.

CHECK $x = -8$

$$\begin{aligned}
2x &= \sqrt{100 - 12x} - 2\\
-16 &\stackrel{?}{=} \sqrt{100 - 12(-8)} - 2\\
-16 &\stackrel{?}{=} \sqrt{196} - 2\\
-16 &\neq 12 \; ✗
\end{aligned}$$

CHECK $x = 3$

$$\begin{aligned}
2x &= \sqrt{100 - 12x} - 2\\
6 &\stackrel{?}{=} \sqrt{100 - 12(3)} - 2\\
6 &\stackrel{?}{=} \sqrt{64} - 2\\
6 &= 6 \; ✓
\end{aligned}$$

One solution checks and the other solution does not. Therefore, the solution is 3.

b. $\sqrt[3]{(x - 5)^2} + 14 = 50$

$$\begin{aligned}
\sqrt[3]{(x - 5)^2} + 14 &= 50 && \text{Original equation}\\
\sqrt[3]{(x - 5)^2} &= 36 && \text{Isolate the radical.}\\
(x - 5)^2 &= 46{,}656 && \text{Raise each side to the third power. (The index is 3.)}\\
x - 5 &= \pm 216 && \text{Take the square root of each side.}\\
x &= 221 \text{ or } -211 && \text{Add 5 to each side.}
\end{aligned}$$

A check of the solutions in the original equation confirms that the solutions are valid.

c. $\sqrt{x - 2} = 5 - \sqrt{15 - x}$

$$\begin{aligned}
\sqrt{x - 2} &= 5 - \sqrt{15 - x} && \text{Original equation}\\
x - 2 &= 25 - 10\sqrt{15 - x} + (15 - x) && \text{Square each side.}\\
2x - 42 &= -10\sqrt{15 - x} && \text{Isolate the radical.}\\
4x^2 - 168x + 1764 &= 100(15 - x) && \text{Square each side.}\\
4x^2 - 168x + 1764 &= 1500 - 100x && \text{Distributive Property}\\
4x^2 - 68x + 264 &= 0 && \text{Combine like terms.}\\
4(x^2 - 17x + 66) &= 0 && \text{Factor.}\\
4(x - 6)(x - 11) &= 0 && \text{Factor.}\\
x - 6 = 0 \quad &\text{or} \quad x - 11 = 0 && \text{Zero Product Property}\\
x = 6 \quad & \quad\quad\; x = 11 && \text{Solve.}
\end{aligned}$$

Watch Out!

Squaring Radical Expressions Take extra care as you square $5 - \sqrt{15 - x}$. While similar to using the FOIL method with binomial expressions, there are some differences. Be sure to account for every term.

A check of the solutions in the original equation confirms that both solutions are valid.

Guided Practice

6A. $3x = 3 + \sqrt{18x - 18}$

6B. $\sqrt[3]{4x + 8} + 3 = 7$

6C. $\sqrt{x + 7} = 3 + \sqrt{2 - x}$

Exercises

● = Step-by-Step Solutions begin on page R29.

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Examples 1 and 2)

1. $f(x) = 5x^2$

2. $g(x) = 8x^5$

3. $h(x) = -x^3$

4. $f(x) = -4x^4$

5. $g(x) = \frac{1}{3}x^9$

6. $f(x) = \frac{5}{8}x^8$

7. $f(x) = -\frac{1}{2}x^7$

8. $g(x) = -\frac{1}{4}x^6$

9. $f(x) = 2x^{-4}$

10. $h(x) = -3x^{-7}$

11. $f(x) = -8x^{-5}$

12. $g(x) = 7x^{-2}$

13. $f(x) = -\frac{2}{5}x^{-9}$

14. $h(x) = \frac{1}{6}x^{-6}$

15. $h(x) = \frac{3}{4}x^{-3}$

16. $f(x) = -\frac{7}{10}x^{-8}$

17. GEOMETRY The volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$, where r is the radius. (Example 1)

a. State the domain and range of the function.

b. Graph the function.

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Example 3)

18. $f(x) = 8x^{\frac{1}{4}}$

19. $f(x) = -6x^{\frac{1}{5}}$

20. $g(x) = -\frac{1}{5}x^{-\frac{1}{3}}$

21. $f(x) = 10x^{-\frac{1}{6}}$

22. $g(x) = -3x^{\frac{5}{8}}$

23. $h(x) = \frac{3}{4}x^{\frac{3}{5}}$

24. $f(x) = -\frac{1}{2}x^{-\frac{3}{4}}$

25. $f(x) = x^{-\frac{2}{3}}$

26. $h(x) = 7x^{\frac{5}{3}}$

27. $h(x) = -4x^{\frac{7}{4}}$

28. $h(x) = -5x^{-\frac{3}{2}}$

29. $h(x) = \frac{2}{3}x^{-\frac{8}{5}}$

Complete each step.

a. **Create a scatter plot of the data.**

b. **Determine a power function to model the data.**

c. **Calculate the value of each model at $x = 30$.** (Example 4)

30.

x	y
1	4
2	22
3	85
4	190
5	370
6	650
7	1000
8	1500

31.

x	y
1	1
2	32
3	360
4	2000
5	7800
6	25,000
7	60,000
8	130,000

32. CLIFF DIVING In the sport of cliff diving, competitors perform three dives from a height of 28 meters. Judges award divers a score from 0 to 10 points based on degree of difficulty, take-off, positions, and water entrance. The table shows the speed of a diver at various distances in the dive. (Example 4)

Distance (m)	Speed (m/s)
4	8.85
8	12.52
12	15.34
16	17.71
20	19.80
24	21.69
28	23.43

a. Create a scatter plot of the data.

b. Determine a power function to model the data.

c. Use the function to predict the speed at which a diver would enter the water from a cliff dive of 30 meters.

33. WEATHER The wind chill temperature is the apparent temperature felt on exposed skin, taking into account the effect of the wind. The table shows the wind chill temperature produced at winds of various speeds when the actual temperature is 50°F. (Example 4)

Wind Speed (mph)	Wind Chill (°F)
5	48.22
10	46.04
15	44.64
20	43.60
25	42.76
30	42.04
35	41.43
40	40.88

a. Create a scatter plot of the data.

b. Determine a power function to model the data.

c. Use the function to predict the wind chill temperature when the wind speed is 65 miles per hour.

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Example 5)

34. $f(x) = 3\sqrt{6 + 3x}$

35. $g(x) = -2\sqrt[5]{1024 + 8x}$

36. $f(x) = -\frac{3}{8}\sqrt[6]{16x + 48} - 3$

㊲ $h(x) = 4 + \sqrt{7x - 12}$

38. $g(x) = \sqrt{(1 - 4x)^3} - 16$

39. $f(x) = -\sqrt[3]{(25x - 7)^2} - 49$

40. $h(x) = \frac{1}{2}\sqrt[3]{27 - 2x} - 8$

41. $g(x) = \sqrt{22 - x} - \sqrt{3x - 3}$

42. FLUID MECHANICS The velocity of the water flowing through a hose with a nozzle can be modeled using $V(P) = 12.1\sqrt{P}$, where V is the velocity in feet per second and P is the pressure in pounds per square inch. (Example 5)

a. Graph the velocity through a nozzle as a function of pressure.

b. Describe the domain, range, end behavior, and continuity of the function and determine where it is increasing or decreasing.

43. AGRICULTURAL SCIENCE The net energy NE_m required to maintain the body weight of beef cattle, in megacalories (Mcal) per day, is estimated by the formula $NE_m = 0.077\sqrt[4]{m^3}$, where m is the animal's mass in kilograms. One megacalorie is equal to one million calories. Example 5

a. Find the net energy per day required to maintain a 400-kilogram steer.

b. If 0.96 megacalorie of energy is provided per pound of whole grain corn, how much corn does a 400-kilogram steer need to consume daily to maintain its body weight?

Solve each **equation.** Example 6

44. $4 = \sqrt{-6 - 2x} + \sqrt{31 - 3x}$
45. $0.5x = \sqrt{4 - 3x} + 2$
46. $-3 = \sqrt{22 - x} - \sqrt{3x - 3}$
47. $\sqrt{(2x - 5)^3} - 10 = 17$
48. $\sqrt{(4x + 164)^3} + 36 = 100$
49. $x = \sqrt{2x - 4} + 2$
50. $7 + \sqrt{(-36 - 5x)^5} = 250$
51. $x = 5 + \sqrt{x + 1}$
52. $\sqrt{6x - 11} + 4 = \sqrt{12x + 1}$
53. $\sqrt{4x - 40} = -20$
54. $\sqrt{x + 2} - 1 = \sqrt{-2 - 2x}$
55. $7 + \sqrt[5]{1054 - 3x} = 11$

Determine whether each function is a monomial function given that *a* and *b* are positive integers. Explain your reasoning.

56. $y = \frac{5}{b}x^{4a}$
57. $G(x) = -2ax^4$
58. $F(b) = 3ab^{5x}$
59. $y = \frac{7}{3}t^{ab}$
60. $H(t) = \frac{1}{ab}t^{\frac{4b}{2}}$
61. $y = 4abx^{-2}$

62. CHEMISTRY The function $r = R_0(A)^{\frac{1}{3}}$ can be used to approximate the nuclear radius of an element based on its molecular mass, where r is length of the radius in meters, R_0 is a constant (about 1.2×10^{-15} meter), and A is the molecular mass.

Element	Molecular Mass
Carbon (C)	12.0
Helium (H)	4.0
Iodine (I)	126.9
Lead (Pb)	207.2
Sodium (Na)	?
Sulfur (S)	32.1

a. If the nuclear radius of sodium is about 3.412×10^{-15} meter, what is its molecular mass?

b. The approximate nuclear radius of an element is 6.030×10^{-15} meter. Identify the element.

c. The ratio of the molecular masses of two elements is 27:8. What is the ratio of their nuclear radii?

Solve each inequality.

63. $\sqrt[5]{1040 + 8x} \geq 4$
64. $\sqrt[13]{41 - 7x} \geq -1$
65. $(1 - 4x)^{\frac{3}{2}} \geq 125$
66. $\sqrt{6 + 3x} \leq 9$
67. $(19 - 4x)^{\frac{5}{3}} - 12 \leq -13$
68. $(2x - 68)^{\frac{2}{3}} \geq 64$

(69) CHEMISTRY Boyle's Law states that, at constant temperature, the pressure of a gas is inversely proportional to its volume. The results of an experiment to explore Boyle's Law are shown.

Volume (liters)	Pressure (atmospheres)
1.0	3.65
1.5	2.41
2.0	1.79
2.5	1.46
3.0	1.21
3.5	1.02
4.0	0.92

a. Create a scatter plot of the data.

b. Determine a power function to model the pressure P as a function of volume v.

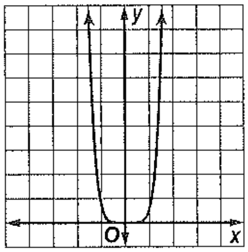
c. Based on the information provided in the problem statement, does the function you determined in part **b** make sense? Explain.

d. Use the model to predict the pressure of the gas if the volume is 3.25 liters.

e. Use the model to predict the pressure of the gas if the volume is 6 liters.

Without using a calculator, match each graph with the appropriate function.

70.



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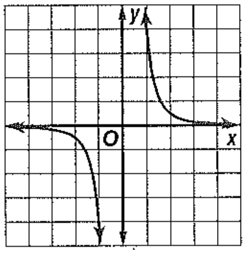

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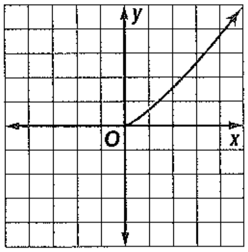

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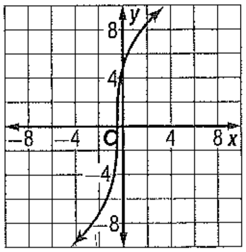


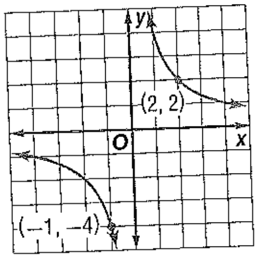
a. $f(x) = \frac{1}{2}\sqrt[4]{3x^5}$
b. $g(x) = \frac{2}{3}x^6$
c. $h(x) = 4x^{-3}$
d. $p(x) = 5\sqrt[3]{2x + 1}$

74. ELECTRICITY The voltage used by an electrical device such as a DVD player can be calculated using $V = \sqrt{PR}$, where V is the voltage in volts, P is the power in watts, and R is the resistance in ohms. The function $I = \sqrt{\frac{P}{R}}$ can be used to calculate the current, where I is the current in amps.

- **a.** If a lamp uses 120 volts and has a resistance of 11 ohms, what is the power consumption of the lamp?
- **b.** If a DVD player has a current of 10 amps and consumes 1200 watts of power, what is the resistance of the DVD player?
- **c.** Ohm's Law expresses voltage in terms of current and resistance. Use the equations given above to write Ohm's Law using voltage, resistance, and amperage.

Use the points provided to determine the power function represented by the graph.

(75)



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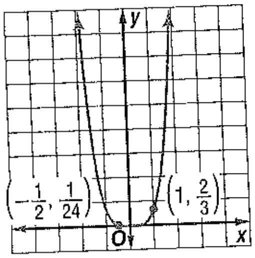

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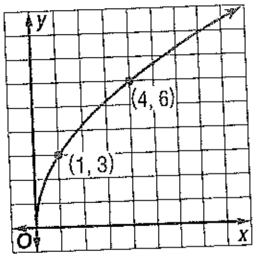

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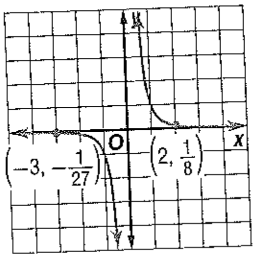
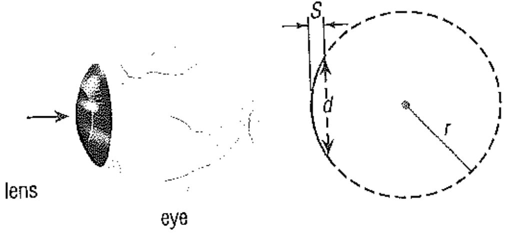


79. OPTICS A contact lens with the appropriate depth ensures proper fit and oxygen permeation. The depth of a lens can be calculated using the formula $S = r - \sqrt{r^2 - \left(\frac{d}{2}\right)^2}$, where S is the depth, r is the radius of curvature, and d is the diameter, with all units in millimeters.



- **a.** If the depth of the contact lens is 1.15 millimeters and the radius of curvature is 7.50 millimeters, what is the diameter of the contact lens?
- **b.** If the depth of the contact lens is increased by 0.1 millimeter and the diameter of the lens is 8.2 millimeters, what radius of curvature would be required?
- **c.** If the radius of curvature remains constant, does the depth of the contact lens increase or decrease as the diameter increases?

80. MULTIPLE REPRESENTATIONS In this problem, you will investigate the average rates of change of power functions.

- **a. GRAPHICAL** For power functions of the form $f(x) = x^n$, graph a function for two values of n such that $0 < n < 1$, $n = 1$, and two values of n such that $n > 1$.
- **b. TABULAR** Copy and complete the table, using your graphs from part **a** to analyze the average rates of change of the functions as x approaches infinity. Describe this rate as *increasing*, *constant*, or *decreasing*.

n	$f(x)$	Average Rate of Change as $x \to \infty$
$0 < n < 1$		
$n = 1$		
$n > 1$		

- **c. VERBAL** Make a conjecture about the average rate of change of a power function as x approaches infinity for the intervals $0 < n < 1$, $n = 1$, and $n > 1$.

H.O.T. Problems Use Higher-Order Thinking Skills

81. CHALLENGE Show that $\sqrt{\frac{8^n \cdot 2^7}{4^{-n}}} = 2^{2n+3}\sqrt{2^{n+1}}$.

82. REASONING Consider $y = 2^x$.

- **a.** Describe the value of y if $x < 0$.
- **b.** Describe the value of y if $0 < x < 1$.
- **c.** Describe the value of y if $x > 1$.
- **d.** Write a conjecture about the relationship between the value of the base and the value of the power if the exponent is greater than or less than 1. Justify your answer.

83. PREWRITE Your senior project is to tutor an underclassman for four sessions on power and radical functions. Make a plan for writing that addresses purpose and audience, and has a controlling idea, logical sequence, and time frame for completion.

84. REASONING Given $f(x) = x^{\frac{a}{b}}$, where a and b are integers with no common factors, determine whether each statement is *true* or *false*. Explain.

- **a.** If the value of b is even and the value of a is odd, then the function is undefined for $x < 0$.
- **b.** If the value of a is even and the value of b is odd, then the function is undefined for $x < 0$.
- **c.** If the value of a is 1, then the function is defined for all x.

85. REASONING Consider $f(x) = x^{\frac{1}{n}} + 5$. How would you expect the graph of the function to change as n increases if n is odd and greater than or equal to 3?

86. WRITING IN MATH Use words, graphs, tables, and equatior to show the relationship between functions in exponenti form and in radical form.

Spiral Review

87. FINANCE If you deposit \$1000 in a savings account with an interest rate of r compounded annually, then the balance in the account after 3 years is given by $B(r) = 1000(1 + r)^3$, where r is written as a decimal. (Lesson 1-7)

a. Find a formula for the interest rate r required to achieve a balance of B in the account after 3 years.

b. What interest rate will yield a balance of \$1100 after 3 years?

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. State the domain of each new function. (Lesson 1-6)

88. $f(x) = x^2 - 2x$
$g(x) = x + 9$

89. $f(x) = \frac{x}{x + 1}$
$g(x) = x^2 - 1$

90. $f(x) = \frac{3}{x - 7}$
$g(x) = x^2 + 5x$

Use the graph of $f(x)$ to graph $g(x) = |f(x)|$ and $h(x) = f(|x|)$. (Lesson 1-5)

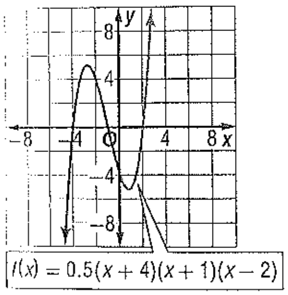
91. $f(x) = -4x + 2$

92. $f(x) = \sqrt{x + 3} - 6$

93. $f(x) = x^2 - 3x - 10$

Use the graph of each function to estimate intervals to the nearest 0.5 unit on which the function is increasing, decreasing, or constant. Support the answer numerically. (Lesson 1-4)

94.



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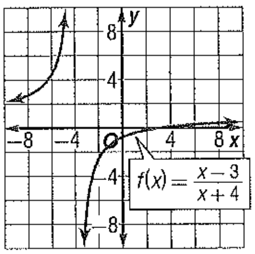

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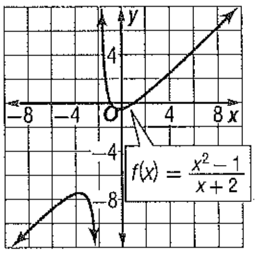


Simplify. (Lesson 0-2)

97. $\frac{\frac{1}{2} + \sqrt{3}i}{1 - \sqrt{2}i}$

98. $\frac{2 - \sqrt{2}i}{3 + \sqrt{6}i}$

99. $\frac{(1 + i)^2}{(-3 + 2i)^2}$

Skills Review for Standardized Tests

100. **SAT/ACT** If m and n are both positive, then which of the following is equivalent to $\frac{2m\sqrt{18n}}{m\sqrt{2}}$?

A $3m\sqrt{n}$
B $6m\sqrt{n}$
C $4\sqrt{n}$
D $6\sqrt{n}$
E $8\sqrt{n}$

101. **REVIEW** If $f(x, y) = x^2y^3$ and $f(a, b) = 10$, what is the value of $f(2a, 2b)$?

F 50
G 100
H 160
J 320
K 640

102. REVIEW The number of minutes m it takes c children to eat p pieces of pizza varies directly as the number of pieces of pizza and inversely as the number of children. If it takes 5 children 30 minutes to eat 10 pieces of pizza, how many minutes should it take 15 children to eat 50 pieces of pizza?

A 30
B 40
C 50
D 60

103. If $\sqrt[3]{5m + 2} = 3$, then $m = ?$

F 3
G 4
H 5
J 6

2-2 Polynomial Functions

Then

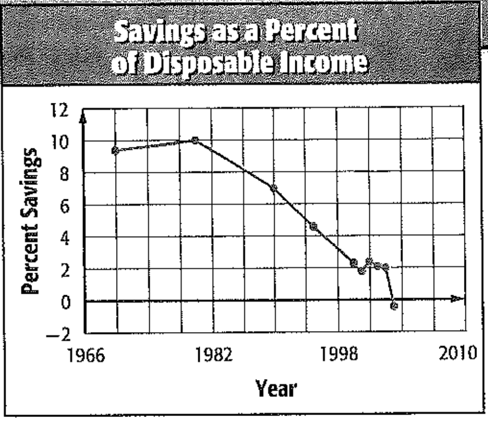
- You analyzed graphs of functions.

Now

1. Graph polynomial functions.
2. Model real-world data with polynomial functions.

Why?

- The scatter plot shows total personal savings as a percent of disposable income in the United States. Often data with multiple relative extrema are best modeled by a polynomial function.


Savings as a Percent of Disposable Income

New Vocabulary

polynomial function
polynomial function of degree n
leading coefficient
leading-term test
quartic function
turning point
quadratic form
repeated zero
multiplicity

1 Graph Polynomial Functions

In Lesson 2-1, you learned about the basic characteristics of monomial functions. Monomial functions are the most basic polynomial functions. The sums and differences of monomial functions form other types of **polynomial functions.**

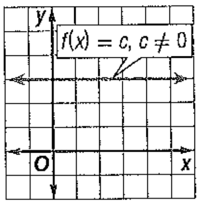
Let n be a nonnegative integer and let $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ be real numbers with $a_n \neq 0$. Then the function given by

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

is called a **polynomial function of degree** n. The **leading coefficient** of a polynomial function is the coefficient of the variable with the greatest exponent. The leading coefficient of $f(x)$ is a_n.

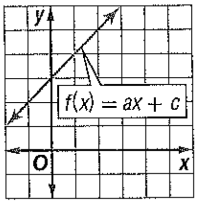
You are already familiar with the following polynomial functions.

Constant Functions



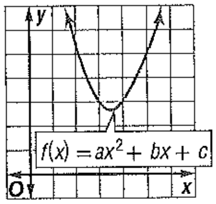
Degree: 0

Linear Functions



Degree: 1

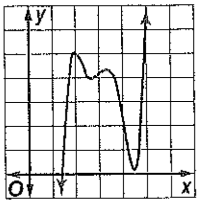
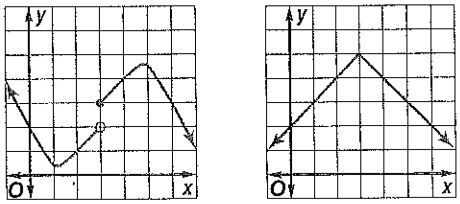
Quadratic Functions



Degree: 2

The zero function is a constant function with no degree. The graphs of polynomial functions share certain characteristics.

Graphs of Polynomial Functions

Example	Nonexamples
 Polynomial functions are defined and continuous for all real numbers and have smooth, rounded turns.	 Graphs of polynomial functions do not have breaks, holes, gaps, or sharp corners.

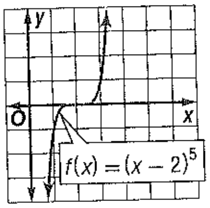
Recall that the graphs of even-degree, non-constant monomial functions resemble the graph of $f(x) = x^2$, while the graphs of odd-degree monomial functions resemble the graph of $f(x) = x^3$. You can use the basic shapes and characteristics of even- and odd-degree monomial functions and what you learned in Lesson 1-5 about transformations to transform graphs of monomial functions.

Example 1 Graph Transformations of Monomial Functions

Graph each function.

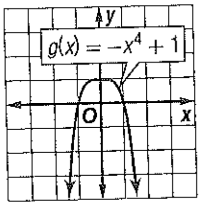
a. $f(x) = (x - 2)^5$

This is an odd-degree function, so its graph is similar to the graph of $y = x^3$. The graph of $f(x) = (x - 2)^5$ is the graph of $y = x^5$ translated 2 units to the right.



b. $g(x) = -x^4 + 1$

This is an even-degree function, so its graph is similar to the graph of $y = x^2$. The graph of $g(x) = -x^4 + 1$ is the graph of $y = x^4$ reflected in the x-axis and translated 1 unit up.



GuidedPractice

1A. $f(x) = 4 - x^3$

1B. $g(x) = (x + 7)^4$

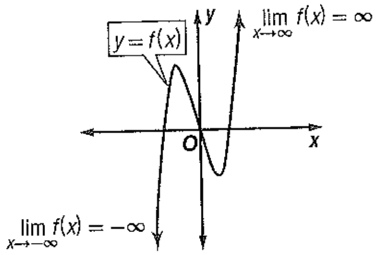
In Lesson 1-3, you learned that the end behavior of a function describes how the function behaves, rising or falling, at either end of its graph. As $x \to -\infty$ and $x \to \infty$, the end behavior of any polynomial function is determined by its leading term. The **leading term test** uses the power and coefficient of this term to determine polynomial end behavior.

KeyConcept Leading Term Test for Polynomial End Behavior

The end behavior of any non-constant polynomial function $f(x) = a_nx^n + \cdots + a_1x + a_0$ can be described in one of the following four ways, as determined by the degree n of the polynomial and its leading coefficient a_n.

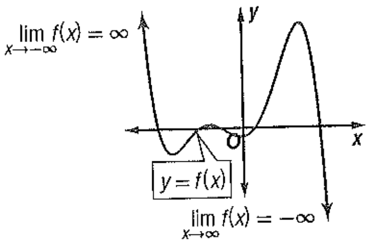
n odd, a_n positive

$\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$



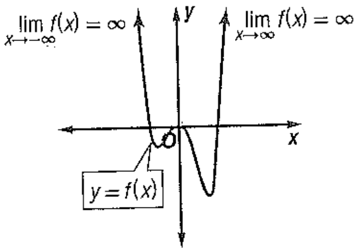
n odd, a_n negative

$\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$



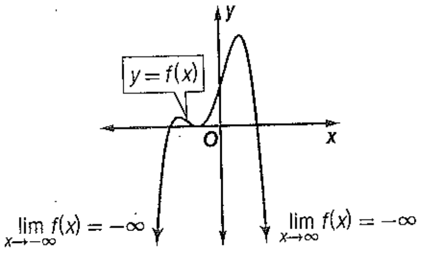
n even, a_n positive

$\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$



n even, a_n negative

$\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$

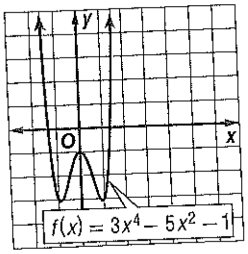


Example 2 Apply the Leading Term Test

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test.

a. $f(x) = 3x^4 - 5x^2 - 1$

The degree is 4, and the leading coefficient is 3. Because the degree is even and the leading coefficient is positive, $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$.

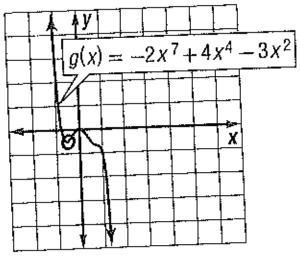


Watch Out!

Standard Form The leading term of a polynomial function is not necessarily the first term of a polynomial. However, the leading term is *always* the first term of a polynomial when the polynomial is written in standard form. Recall that a polynomial is written in standard form if its terms are written in descending order of exponents.

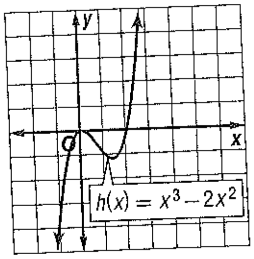
b. $g(x) = -3x^2 - 2x^7 + 4x^4$

Write in standard form as $g(x) = -2x^7 + 4x^4 - 3x^2$. The degree is 7, and the leading coefficient is −2. Because the degree is odd and the leading coefficient is negative, $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$.



c. $h(x) = x^3 - 2x^2$

The degree is 3, and the leading coefficient is 1. Because the degree is odd and the leading coefficient is positive, $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$.

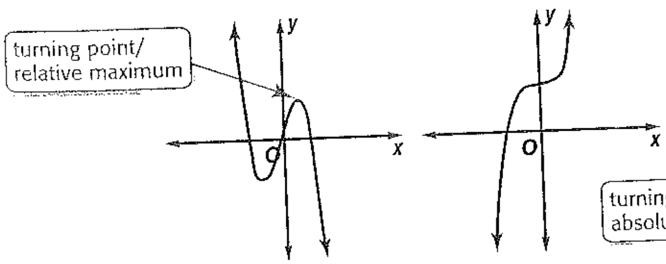


Guided Practice

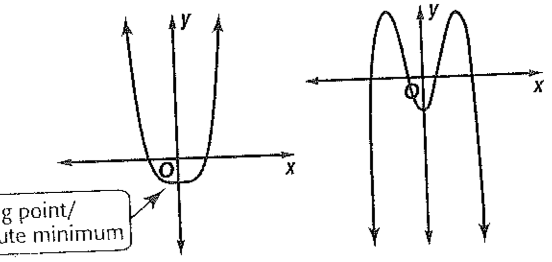
2A. $g(x) = 4x^5 - 8x^3 + 20$

2B. $h(x) = -2x^6 + 11x^4 + 2x^2$

Consider the shapes of a few typical third-degree polynomial or cubic functions and fourth-degree polynomial or **quartic functions** shown.



Typical Cubic Functions



Typical Quartic Functions

Observe the number of x-intercepts for each graph. Because an x-intercept corresponds to a real zero of the function, you can see that cubic functions have at most 3 zeros and quartic functions have at most 4 zeros.

Turning points indicate where the graph of a function changes from increasing to decreasing, and vice versa. Maxima and minima are also located at turning points. Notice that cubic functions have at most 2 turning points, and quartic functions have at most 3 turning points. These observations can be generalized as follows and shown to be true for any polynomial function.

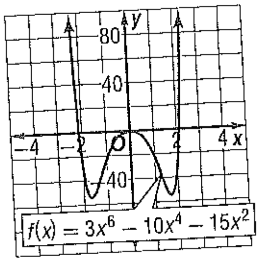
StudyTip

Look Back Recall from Lesson 1-2 that the *x*-intercepts of the graph of a function are also called the zeros of a function. The solutions of the corresponding equation are called the roots of the equation.

KeyConcept Zeros and Turning Points of Polynomial Functions

A polynomial function f of degree $n \geq 1$ has at most n distinct real zeros and at most $n - 1$ turning points.

Example Let $f(x) = 3x^6 - 10x^4 - 15x^2$. Then f has at most 6 distinct real zeros and at most 5 turning points. The graph of f suggests that the function has 3 real zeros and 3 turning points.



Recall that if f is a polynomial function and c is an x-intercept of the graph of f, then it is equivalent to say that:

- c is a zero of f,
- $x = c$ is a solution of the equation $f(x) = 0$, and
- $(x - c)$ is a factor of the polynomial $f(x)$.

You can find the zeros of some polynomial functions using the same factoring techniques you used to solve quadratic equations.

Example 3 Zeros of a Polynomial Function

State the number of possible real zeros and turning points of $f(x) = x^3 - 5x^2 + 6x$. Then determine all of the real zeros by factoring.

The degree of the function is 3, so f has at most 3 distinct real zeros and at most 3 − 1 or 2 turning points. To find the real zeros, solve the related equation $f(x) = 0$ by factoring.

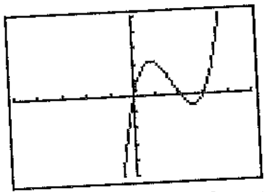
$x^3 - 5x^2 + 6x = 0$ Set $f(x)$ equal to 0.

$x(x^2 - 5x + 6) = 0$ Factor the greatest common factor x.

$x(x - 2)(x - 3) = 0$ Factor completely.

So, f has three distinct real zeros, 0, 2, and 3. This is consistent with a cubic function having at most 3 distinct real zeros.

CHECK You can use a graphing calculator to graph $f(x) = x^3 - 5x^2 + 6x$ and confirm these zeros. Additionally, you can see that the graph has 2 turning points, which is consistent with cubic functions having at most 2 turning points.



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GuidedPractice

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.

3A. $f(x) = x^3 - 6x^2 - 27x$

3B. $f(x) = x^4 - 8x^2 + 15$

StudyTip

Look Back To review techniques for solving quadratic equations, see Lesson 0-3.

In some cases, a polynomial function can be factored using quadratic techniques if it has *quadratic form*.

KeyConcept Quadratic Form

Words	A polynomial expression in x is in **quadratic form** if it is written as $au^2 + bu + c$ for any numbers a, b, and c, $a \neq 0$, where u is some expression in x.
Symbols	$x^4 - 5x^2 - 14$ is in quadratic form because the expression can be written as $(x^2)^2 - 5(x^2) - 14$. If $u = x^2$, then the expression becomes $u^2 - 5u - 14$.

Example 4 Zeros of a Polynomial Function in Quadratic Form

State the number of possible real zeros and turning points for $g(x) = x^4 - 3x^2 - 4$. Then determine all of the real zeros by factoring.

The degree of the function is 4, so g has at most 4 distinct real zeros and at most $4 - 1$ or 3 turning points. This function is in quadratic form because $x^4 - 3x^2 - 4 = (x^2)^2 - 3(x^2) - 4$. Let $u = x^2$.

$$(x^2)^2 - 3(x^2) - 4 = 0 \qquad \text{Set } g(x) \text{ equal to 0.}$$

$$u^2 - 3u - 4 = 0 \qquad \text{Substitute } u \text{ for } x^2.$$

$$(u + 1)(u - 4) = 0 \qquad \text{Factor the quadratic expression.}$$

$$(x^2 + 1)(x^2 - 4) = 0 \qquad \text{Substitute } x^2 \text{ for } u.$$

$$(x^2 + 1)(x + 2)(x - 2) = 0 \qquad \text{Factor completely.}$$

$$x^2 + 1 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Zero Product Property}$$

$$x = \pm\sqrt{-1} \qquad x = -2 \qquad x = 2 \qquad \text{Solve for } x.$$

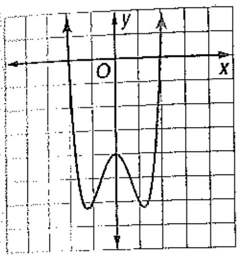


Figure 2.2.1

Because $\pm\sqrt{-1}$ are not real zeros, g has two distinct real zeros, −2 and 2. This is consistent with a quartic function. The graph of $g(x) = x^4 - 3x^2 - 4$ in Figure 2.2.1 confirms this. Notice that there are 3 turning points, which is also consistent with a quartic function.

Guided Practice

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.

4A. $g(x) = x^4 - 9x^2 + 18$

4B. $h(x) = x^5 - 6x^3 - 16x$

If a factor $(x - c)$ occurs more than once in the completely factored form of $f(x)$, then its related zero c is called a **repeated zero.** When the zero occurs an even number of times, the graph will be tangent to the x-axis at that point. When the zero occurs an odd number of times, the graph will cross the x-axis at that point. A graph is tangent to an axis when it touches the axis at that point, but does not cross it.

Example 5 Polynomial Function with Repeated Zeros

State the number of possible real zeros and turning points of $h(x) = -x^4 - x^3 + 2x^2$. Then determine all of the real zeros by factoring.

The degree of the function is 4, so h has at most 4 distinct real zeros and at most $4 - 1$ or 3 turning points. Find the real zeros.

$$-x^4 - x^3 + 2x^2 = 0 \qquad \text{Set } h(x) \text{ equal to 0}$$

$$-x^2(x^2 + x - 2) = 0 \qquad \text{Factor the greatest common factor, } -x^2.$$

$$-x^2(x - 1)(x + 2) = 0 \qquad \text{Factor completely.}$$

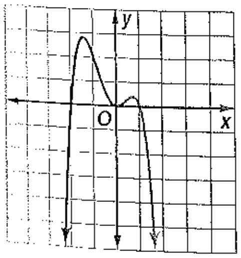


Figure 2.2.2

The expression above has 4 factors, but solving for x yields only 3 distinct real zeros, 0, 1, and −2. Of the zeros, 0 occurs twice.

The graph of $h(x) = -x^4 - x^3 + 2x^2$ shown in Figure 2.2.2 confirms these zeros and shows that h has three turning points. Notice that at $x = 1$ and $x = -2$, the graph crosses the x-axis, but at $x = 0$, the graph is tangent to the x-axis.

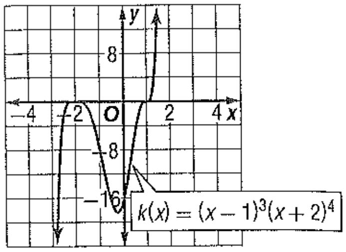
Guided Practice

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.

5A. $g(x) = -2x^3 - 4x^2 + 16x$

5B. $f(x) = 3x^5 - 18x^4 + 27x^3$

In $h(x) = -x^2(x - 1)(x + 2)$ from Example 5, the zero $x = 0$ occurs 2 times. In $k(x) = (x - 1)^3(x + 2)^4$, the zero $x = 1$ occurs 3 times, while $x = -2$ occurs 4 times. Notice that in the graph of k shown, the curve crosses the x-axis at $x = 1$ but not at $x = -2$. These observations can be generalized as follows and shown to be true for all polynomial functions.



StudyTip

Nonrepeated Zeros A nonrepeated zero can be thought of as having a multiplicity of 1 or odd multiplicity. A graph crosses the x-axis and has a sign change at every nonrepeated zero.

KeyConcept Repeated Zeros of Polynomial Functions

If $(x - c)^m$ is the highest power of $(x - c)$ that is a factor of polynomial function f, then c is a zero of **multiplicity** m of f, where m is a natural number.

- If a zero c has odd multiplicity, then the graph of f crosses the x-axis at $x = c$ and the value of $f(x)$ changes signs at $x = c$.
- If a zero c has even multiplicity, then the graph of f is tangent to the x-axis at $x = c$ and the value of $f(x)$ does not change signs at $x = c$.

You now have several tests and tools to aid you in graphing polynomial functions.

Example 6 Graph a Polynomial Function

For $f(x) = x(2x + 3)(x - 1)^2$, (a) apply the leading-term test, (b) determine the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function.

a. The product $x(2x + 3)(x - 1)^2$ has a leading term of $x(2x)(x)^2$ or $2x^4$, so f has degree 4 and leading coefficient 2. Because the degree is even and the leading coefficient is positive, $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$.

b. The distinct real zeros are 0, −1.5, and 1. The zero at 1 has multiplicity 2.

c. Choose x-values that fall in the intervals determined by the zeros of the function.

Interval	x-value in Interval	f(x)	(x, f(x))
$(-\infty, -1.5)$	−2	$f(-2) = 18$	(−2, 18)
$(-1.5, 0)$	−1	$f(-1) = -4$	(−1, −4)
$(0, 1)$	0.5	$f(0.5) = 0.5$	(0.5, 0.5)
$(1, \infty)$	1.5	$f(1.5) = 2.25$	(1.5, 2.25)

d. Plot the points you found (Figure 2.2.3). The end behavior of the function tells you that the graph eventually rises to the left and to the right. You also know that the graph crosses the x-axis at nonrepeated zeros −1.5 and 0, but does not cross the x-axis at repeated zero 1, because its multiplicity is even. Draw a continuous curve through the points as shown in Figure 2.2.4.

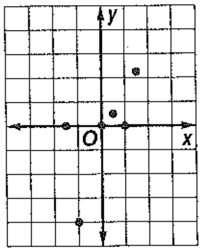


Figure 2.2.3

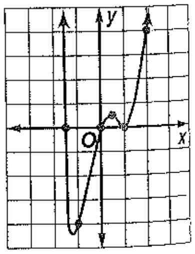


Figure 2.2.4

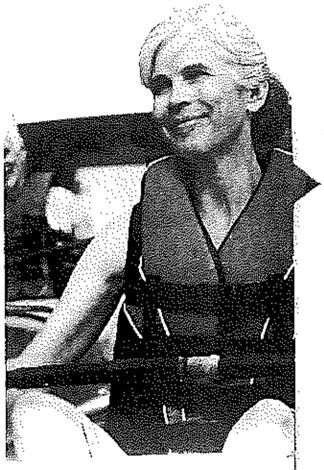
GuidedPractice

For each function, (a) apply the leading-term test, (b) determine the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function.

6A. $f(x) = -2x(x - 4)(3x - 1)^3$

6B. $h(x) = -x^3 + 2x^2 + 8x$

2 Model Data You can use a graphing calculator to model data that exhibit linear, quadratic, cubic, and quartic behavior by first examining the number of turning points suggested by a scatter plot of the data.



Real-WorldLink

A college graduate planning to retire at 65 needs to save an average of $10,000 per year toward retirement.

Source: Monroe Bank

Real-World Example 7 Model Data Using Polynomial Functions

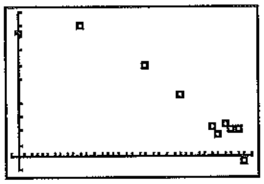
SAVINGS Refer to the beginning of the lesson. The average personal savings as a percent of disposable income in the United States is given in the table.

Year	1970	1980	1990	1995	2000	2001	2002	2003	2004	2005
% Savings	9.4	10.0	7.0	4.6	2.3	1.8	2.4	2.1	2.0	−0.4

Source: U.S. Department of Commerce

a. Create a scatter plot of the data and determine the type of polynomial function that could be used to represent the data.

Enter the data using the list feature of a graphing calculator. Let L1 be the number of years since 1970. Then create a scatter plot of the data. The curve of the scatter plot resembles the graph of a quadratic equation, so we will use a quadratic regression.



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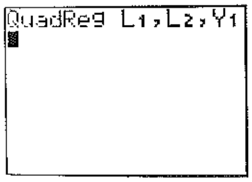
b. Write a polynomial function to model the data set. Round each coefficient to the nearest thousandth, and state the correlation coefficient.

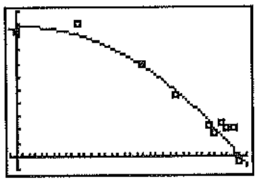
Using the QuadReg tool on a graphing calculator and rounding each coefficient to the nearest thousandth yields $f(x) = -0.009x^2 + 0.033x + 9.744$. The correlation coefficient r^2 for the data is 0.96, which is close to 1, so the model is a good fit.



QuadReg L₁,L₂,Y₁

Figure 2.2.5

We can graph the complete (unrounded) regression by sending it to the Y= menu. If you enter L_1, L_2, and Y_1 after QuadReg, as shown in Figure 2.2.5, the regression equation will be entered into Y_1. Graph this function and the scatter plot in the same viewing window. The function appears to fit the data reasonably well.



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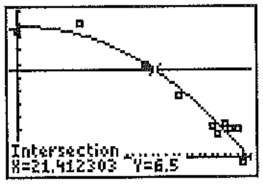
c. Use the model to estimate the percent savings in 1993.

Because 1993 is 23 years after 1970, use the CALC feature on a calculator to find $f(23)$. The value of $f(23)$ is 5.94, so the percent savings in 1993 was about 5.94%.

d. Use the model to determine the approximate year in which the percent savings reached 6.5%.

Graph the line $y = 6.5$ for Y_2. Then use 5: intersect on the CALC menu to find the point of intersection of $y = 6.5$ with $f(x)$. The intersection occurs when $x \approx 21$, so the approximate year in which the percent savings reached 6.5% was about 1970 + 21 or 1991.



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GuidedPractice

7. POPULATION The median age of the U.S. population by year predicted through 2080 is shown.

Year	1900	1930	1960	1990	2020	2050	2080
Median Age	22.9	26.5	29.5	33.0	40.2	42.7	43.9

Source: U.S. Census Bureau

a. Write a polynomial function to model the data. Let L1 be the number of years since 1900.

b. Estimate the median age of the population in 2005.

c. According to your model, in what year did the median age of the population reach 30?

Exercises

○ = Step-by-Step Solutions begin on page R29.

Graph each function. (Example 1)

1. $f(x) = (x + 5)^2$
2. $f(x) = (x - 6)^3$
3. $f(x) = x^4 - 6$
4. $f(x) = x^5 + 7$
5. $f(x) = (2x)^4$
6. $f(x) = (2x)^5 - 16$
7. $f(x) = (x - 3)^4 + 6$
8. $f(x) = (x + 4)^3 - 3$
9. $f(x) = \frac{1}{3}(x - 9)^5$
10. $f(x) = \left(\frac{1}{2}x\right)^3 + 8$

11. WATER If it takes exactly one minute to drain a 10-gallon tank of water, the volume of water remaining in the tank can be approximated by $v(t) = 10(1 - t)^2$, where t is time in minutes, $0 \le t \le 1$. Graph the function. (Example 1)

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test. (Example 2)

12. $f(x) = -5x^7 + 6x^4 + 8$
13. $f(x) = 2x^6 + 4x^5 + 9x^2$
14. $g(x) = 5x^4 + 7x^5 - 9$
15. $g(x) = -7x^3 + 8x^4 - 6x^6$
16. $h(x) = 8x^2 + 5 - 4x^3$
17. $h(x) = 4x^2 + 5x^3 - 2x^5$
18. $f(x) = x(x + 1)(x - 3)$
19. $g(x) = x^2(x + 4)(-2x + 1)$
20. $f(x) = -x(x - 4)(x + 5)$
21. $g(x) = x^3(x + 1)(x^2 - 4)$

22. ORGANIC FOOD The number of acres in the United States used for organic apple production from 2000 to 2005 can be modeled by $a(x) = 43.77x^4 - 498.76x^3 + 1310.2x^2 + 1626.2x + 6821.5$, where $x = 0$ is 2000. (Example 2)
    a. Graph the function using a graphing calculator.
    b. Describe the end behavior of the graph of the function using limits. Explain using the leading term test.

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring. (Examples 3–5)

㉓ $f(x) = x^5 + 3x^4 + 2x^3$

24. $f(x) = x^6 - 8x^5 + 12x^4$
25. $f(x) = x^4 + 4x^2 - 21$
26. $f(x) = x^4 - 4x^3 - 32x^2$
27. $f(x) = x^6 - 6x^3 - 16$
28. $f(x) = 4x^8 + 16x^4 + 12$
29. $f(x) = 9x^6 - 36x^4$
30. $f(x) = 6x^5 - 150x^3$
31. $f(x) = 4x^4 - 4x^3 - 3x^2$
32. $f(x) = 3x^5 + 11x^4 - 20x^3$

For each function, (a) apply the leading-term test, (b) determine the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function. (Example 6)

33. $f(x) = x(x + 4)(x - 1)^2$
34. $f(x) = x^2(x - 4)(x + 2)$
35. $f(x) = -x(x + 3)^2(x - 5)$
36. $f(x) = 2x(x + 5)^2(x - 3)$
37. $f(x) = -x(x - 3)(x + 2)^3$
38. $f(x) = -(x + 2)^2(x - 4)^2$
39. $f(x) = 3x^3 - 3x^2 - 36x$
40. $f(x) = -2x^3 - 4x^2 + 6x$
41. $f(x) = x^4 + x^3 - 20x^2$
42. $f(x) = x^5 + 3x^4 - 10x^3$

43. RESERVOIRS The number of feet below the maximum water level in Wisconsin's Rainbow Reservoir during ten months in 2007 is shown. (Example 7)

Month	Level	Month	Level
January	4	July	9
February	5.5	August	11
March	10	September	16.5
April	9	November	11.5
May	7.5	December	8.5

Source: Wisconsin Valley Improvement Company

    a. Write a model that best relates the water level as a function of the number of months since January.
    b. Use the model to estimate the water level in the reservoir in October.

Use a graphing calculator to write a polynomial function to model each set of data. (Example 7)

44.

x	−3	−2	−1	0	1	2	3
$f(x)$	8.75	7.5	6.25	5	3.75	2.5	1.25

45.

x	5	7	8	10	11	12	15	16
$f(x)$	2	5	6	4	−1	−3	5	9

46.

x	−2.53	−2	−1.5	−1	−0.5	0	0.5	1	1.5
$f(x)$	23	11	7	6	6	5	3	2	4

47.

x	30	35	40	45	50	55	60	65	70	75
$f(x)$	52	41	32	44	61	88	72	59	66	93

48. ELECTRICITY The average retail electricity prices in the U.S. from 1970 to 2005 are shown. Projected prices for 2010 and 2020 are also shown. (Example 7)

Year	Price (¢ / kWh)	Year	Price (¢ / kWh)
1970	6.125	1995	7.5
1974	7	2000	6.625
1980	7.25	2005	6.25
1982	9.625	2010	6.25
1990	8	2020	6.375

Source: Energy Information Administration

    a. Write a model that relates the price as a function of the number of years since 1970.
    b. Use the model to predict the average price of electricity in 2015.
    c. According to the model, during which year was the price 7¢ for the second time?

49. COMPUTERS The numbers of laptops sold each quarter from 2005 to 2007 are shown. Let the first quarter of 2005 be 1, and the fourth quarter of 2007 be 12.

Quarters	Sale (Thousands)
1	423
2	462
3	495
4	634
5	587
6	498
7	798
8	986
9	969
10	891
11	1130
12	1347

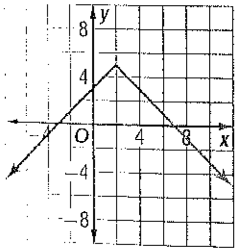
a. Predict the end behavior of a graph of the data as x approaches infinity.

b. Use a graphing calculator to graph and model the data. Is the model a good fit? Explain your reasoning.

c. Describe the end behavior of the graph using limits. Was your prediction accurate? Explain your reasoning.

Determine whether each graph could show a polynomial function. Write *yes* or *no*. If not, explain why not.

50.



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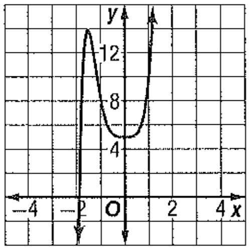

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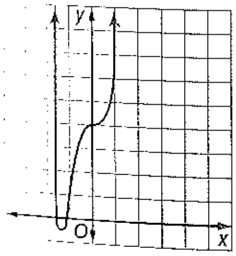

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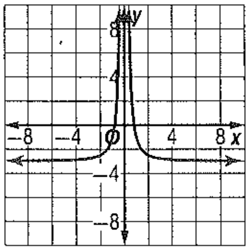


Find a polynomial function of degree n with only the following real zeros. More than one answer is possible.

54. -1; $n = 3$

55. 3; $n = 3$

56. $6, -3$; $n = 4$

57. $-5, 4$; $n = 4$

58. 7; $n = 4$

59. $0, -4$; $n = 5$

60. $2, 1, 4$; $n = 5$

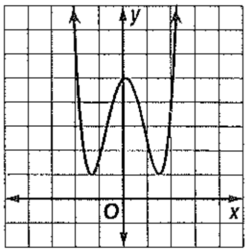
61. $0, 3, -2$; $n = 5$

62. no real zeros; $n = 4$

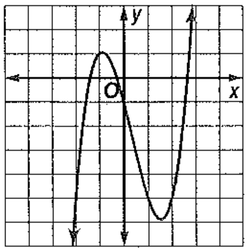
63. no real zeros; $n = 6$

Determine whether the degree n of the polynomial for each graph is *even* or *odd* and whether its leading coefficient a_n is *positive* or *negative*.

64.



65.



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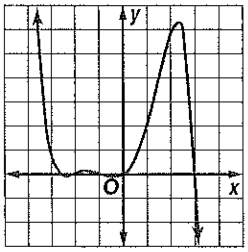

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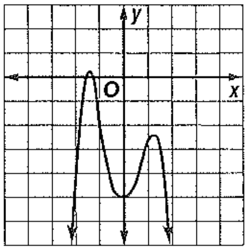


68. MANUFACTURING A company manufactures aluminum containers for energy drinks.

a. Write an equation V that represents the volume of the container.

b. Write a function A in terms of r that represents the surface area of a container with a volume of 15 cubic inches.

c. Use a graphing calculator to determine the minimum possible surface area of the can.

Determine a polynomial function that has each set of zeros. More than one answer is possible.

69. $5, -3, 6$

70. $4, -8, -2$

71. $3, 0, 4, -1, 3$

72. $1, 1, -4, 6, 0$

73. $\frac{3}{4}, -3, -4, -\frac{2}{3}$

74. $-1, -1, 5, 0, \frac{5}{6}$

(75) POPULATION The percent of the United States population living in metropolitan areas has increased.

Year	Percent of Population
1950	56.1
1960	63
1970	68.6
1980	74.8
1990	74.8
2000	79.2

Source: U.S. Census Bureau

a. Write a model that relates the percent as a function of the number of years since 1950.

b. Use the model to predict the percent of the population that will be living in metropolitan areas in 2015.

c. Use the model to predict the year in which 85% of the population will live in metropolitan areas.

Create a function with the following characteristics. Graph the function.

76. degree = 5, 3 real zeros, $\lim_{x \to \infty} = \infty$

77. degree = 6, 4 real zeros, $\lim_{x \to \infty} = -\infty$

78. degree = 5, 2 distinct real zeros, 1 of which has a multiplicity of 2, $\lim_{x \to \infty} = \infty$

79. degree = 6, 3 distinct real zeros, 1 of which has a multiplicity of 2, $\lim_{x \to \infty} = -\infty$

80. WEATHER The temperatures in degrees Celsius from 10 A.M. to 7 P.M. during one day for a city are shown where x is the number of hours since 10 A.M.

Time	Temp.	Time	Temp.
0	4.1	5	10
1	5.7	6	7
2	7.2	7	4.6
3	7.3	8	2.3
4	9.4	9	−0.4

a. Graph the data.

b. Use a graphing calculator to model the data using a polynomial function with a degree of 3.

c. Repeat part **b** using a function with a degree of 4.

d. Which function is a better model? Explain.

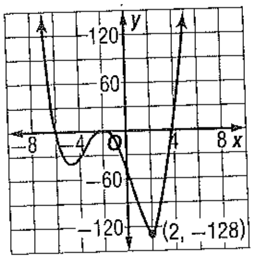
For each of the following graphs:

a. Determine the degree and end behavior.

b. Locate the zeros and their multiplicity. Assume all of the zeros are integral values.

c. Use the given point to determine a function that fits the graph.

81.



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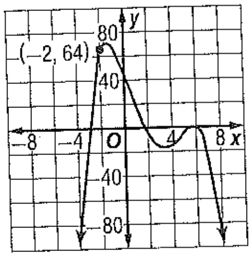

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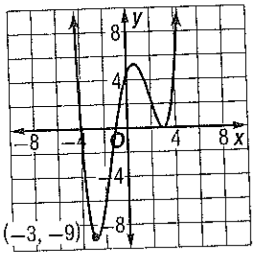

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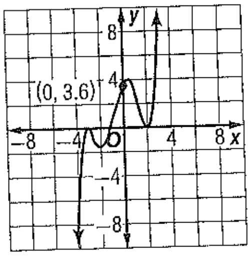


State the number of possible real zeros and turning points of each function. Then find all of the real zeros by factoring.

85. $f(x) = 16x^4 + 72x^2 + 80$

86. $f(x) = -12x^3 - 44x^2 - 40x$

(87) $f(x) = -24x^4 + 24x^3 - 6x^2$

88. $f(x) = x^3 + 6x^2 - 4x - 24$

89. MULTIPLE REPRESENTATIONS In this problem, you will investigate the behavior of combinations of polynomial functions.

a. GRAPHICAL Graph $f(x)$, $g(x)$, and $h(x)$ in each row on the same graphing calculator screen. For each graph, modify the window to observe the behavior both on a large scale and very close to the origin.

$f(x) =$	$g(x) =$	$h(x) =$
$x^2 + x$	x^2	x
$x^3 - x$	x^3	$-x$
$x^3 + x^2$	x^3	x^2

b. ANALYTICAL Describe the behavior of each graph of $f(x)$ in terms of $g(x)$ or $h(x)$ near the origin.

c. ANALYTICAL Describe the behavior of each graph of $f(x)$ in terms of $g(x)$ or $h(x)$ as x approaches ∞ and $-\infty$.

d. VERBAL Predict the behavior of a function that is a combination of two functions a and b such that $f(x) = a + b$, where a is the term of higher degree.

H.O.T. Problems Use Higher-Order Thinking Skills

90. ERROR ANALYSIS Colleen and Martin are modeling the data shown. Colleen thinks the model should be $f(x) = 5.754x^3 + 2.912x^2 - 7.516x + 0.349$. Martin thinks it should be $f(x) = 3.697x^2 + 11.734x - 2.476$. Is either of them correct? Explain your reasoning.

x	$f(x)$	x	$f(x)$
−2	−19	0.5	−2
−1	5	1	1.5
0	0.4	2	43

91. REASONING Can a polynomial function have both an absolute maximum and an absolute minimum? Explain your reasoning.

92. REASONING Explain why the constant function $f(x) = c$, $c \neq 0$, has degree 0, but the zero function $f(x) = 0$ has no degree.

93. CHALLENGE Use factoring by grouping to determine the zeros of $f(x) = x^3 + 5x^2 - x^2 - 5x - 12x - 60$. Explain each step.

94. REASONING How is it possible for more than one function to be represented by the same degree, end behavior, and distinct real zeros? Provide an example to explain your reasoning.

95. REASONING What is the minimum degree of a polynomial function that has an absolute maximum, a relative maximum, and a relative minimum? Explain your reasoning.

96. WRITING IN MATH Explain how you determine the best polynomial function to use when modeling data.

Spiral Review

Solve each equation. (Lesson 2-1)

97. $\sqrt{z+3} = 7$

98. $d + \sqrt{d^2 - 8} = 4$

99. $\sqrt{x-8} = \sqrt{13+x}$

100. REMODELING An installer is replacing the carpet in a 12-foot by 15-foot living room. The new carpet costs \$13.99 per square yard. The formula $f(x) = 9x$ converts square yards to square feet. (Lesson 1-7)

a. Find the inverse $f^{-1}(x)$. What is the significance of $f^{-1}(x)$?

b. How much will the new carpet cost?

Given $f(x) = 2x^2 - 5x + 3$ and $g(x) = 6x + 4$, find each function. (Lesson 1-6)

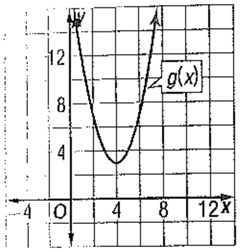
101. $(f + g)(x)$

102. $[f \circ g](x)$

103. $[g \circ f](x)$

Describe how the graphs of $f(x) = x^2$ and $g(x)$ are related. Then write an equation for $g(x)$. (Lesson 1-5)

104.



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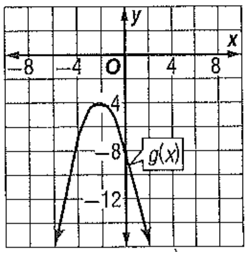

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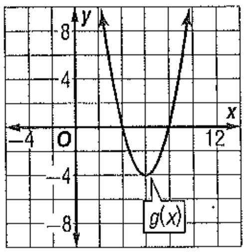


107. BUSINESS A company creates a new product that costs \$25 per item to produce. They hire a marketing analyst to help determine a selling price. After collecting and analyzing data relating selling price s to yearly consumer demand d, the analyst estimates demand for the product using $d = -200s + 15{,}000$. (Lesson 1-4)

a. If yearly profit is the difference between total revenue and production costs, determine a selling price $s \geq 25$, that will maximize the company's yearly profit P. (*Hint*: $P = sd - 25d$)

b. What are the risks of determining a selling price using this method?

The scores for an **exam given in physics class are given.** (Lesson 0-8)

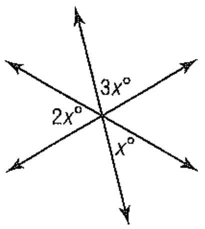
82, 77, 84, 98, 93, 71, 76, 64, 89, 95, 78, 89, 65, 88, 54,
96, 87, 92, 80, 85, 93, 89, 55, 62, 79, 90, 86, 75, 99, 62

108. Make a box-and-whisker plot of the test.

109. What is the standard deviation of the test scores?

Skills Review for Standardized Tests

110. **SAT/ACT** The figure shows the intersection of three lines. The figure is not drawn to scale.



$x =$

A 16

B 20

C 30

D 60

E 90

111. Over the domain $2 < x \leq 3$, which of the following functions contains the greatest values of y?

F $y = \dfrac{x+3}{x-2}$

G $y = \dfrac{x-5}{x+1}$

H $y = x^2 - 3$

J $y = 2x$

112. MULTIPLE CHOICE Which of the following equations represents the result of shifting the parent function $y = x^3$ up 4 units and right 5 units?

A $y + 4 = (x + 5)^3$

B $y - 4 = (x + 5)^3$

C $y + 4 = (x - 5)^3$

D $y - 4 = (x - 5)^3$

113. REVIEW Which of the following describes the numbers in the domain of $h(x) = \dfrac{\sqrt{2x-3}}{x-5}$?

F $x \neq 5$

G $x \geq \frac{3}{2}$

H $x \geq \frac{3}{2}, x \neq 5$

J $x \neq \frac{3}{2}$

LESSON

2-3 The Remainder and Factor Theorems

:: Then

- You factored quadratic expressions to solve equations.

:: Now

1. Divide polynomials using long division and synthetic division.
2. Use the Remainder and Factor Theorems.

:: Why?

- The redwood trees of Redwood National Park in California are the oldest living species in the world. The trees can grow up to 350 feet and can live up to 2000 years. Synthetic division can be used to determine the height of one of the trees during a particular year.

NewVocabulary

synthetic division
depressed polynomial
synthetic substitution

1 Divide Polynomials

Consider the polynomial function $f(x) = 6x^3 - 25x^2 + 18x + 9$. If you know that f has a zero at $x = 3$, then you also know that $(x - 3)$ is a factor of $f(x)$. Because $f(x)$ is a third-degree polynomial, you know that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 3) \cdot q(x).$$

This implies that $q(x)$ can be found by dividing $6x^3 - 25x^2 + 18x + 9$ by $(x - 3)$ because

$$q(x) = \frac{f(x)}{x - 3}, \text{ if } x \neq 3.$$

To divide polynomials, we can use an algorithm similar to that of long division with integers.

Example 1 Use Long Division to Factor Polynomials

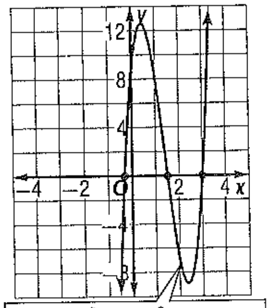
Factor $6x^3 - 25x^2 + 18x + 9$ completely using long division if $(x - 3)$ is a factor.

$$\begin{array}{rll}
 & 6x^2 - 7x - 3 & \\
x - 3 \overline{)} & 6x^3 - 25x^2 + 18x + 9 & \\
(-) & \underline{6x^3 - 18x^2} & \text{Multiply divisor by } 6x^2 \text{ because } \frac{6x^3}{x} = 6x^2. \\
 & -7x^2 + 18x & \text{Subtract and bring down next term.} \\
(-) & \underline{-7x^2 + 21x} & \text{Multiply divisor by } -7x \text{ because } \frac{-7x^2}{x} = -7x. \\
 & -3x + 9 & \text{Subtract and bring down next term.} \\
(-) & \underline{-3x + 9} & \text{Multiply divisor by } -3 \text{ because } \frac{-3x}{x} = -3. \\
 & 0 & \text{Subtract. Notice that the remainder is 0.}
\end{array}$$

From this division, you can write $6x^3 - 25x^2 + 18x + 9 = (x - 3)(6x^2 - 7x - 3)$.

Factoring the quadratic expression yields $6x^3 - 25x^2 + 18x + 9 = (x - 3)(2x - 3)(3x + 1)$.

So, the zeros of the polynomial function $f(x) = 6x^3 - 25x^2 + 18x + 9$ are 3, $\frac{3}{2}$, and $-\frac{1}{3}$. The x-intercepts of the graph of $f(x)$ shown support this conclusion.



$f(x) = 6x^3 - 25x^2 + 18x + 9$

GuidedPractice

Factor each polynomial completely using the given factor and long division.

1A. $x^3 + 7x^2 + 4x - 12; x + 6$

1B. $6x^3 - 2x^2 - 16x - 8; 2x - 4$

Long division of polynomials can result in a zero remainder, as in Example 1, or a nonzero remainder, as in the example below. Notice that just as with integer long division, the result of polynomial division is expressed using the quotient, remainder, and divisor.

$$\begin{array}{r l}
x+3 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow x+2\overline{)x^2+5x-4} & \leftarrow \text{Dividend} \\
(-)\ \underline{x^2+2x} & \\
3x-4 & \\
\underline{(-)\ 3x+6} & \\
\text{Remainder} \rightarrow \mathbf{-10} &
\end{array}$$

$$\begin{array}{l}\text{Dividend} \rightarrow \\ \text{Divisor} \rightarrow\end{array} \frac{x^2+5x-4}{x+2} = \overset{\text{Quotient}}{x+3} + \frac{\overset{\text{Remainder}}{-10}}{\underset{\text{Divisor}}{x+2}}, \underset{\text{Excluded value}}{x \neq -2}$$

StudyTip

Proper vs. Improper A rational expression is considered improper if the degree of the numerator is greater than or equal to the degree of the denominator. So in the division algorithm, $\frac{f(x)}{d(x)}$ is an *improper* rational expression, while $\frac{r(x)}{d(x)}$ is a *proper* rational expression.

Recall that a dividend can be expressed in terms of the divisor, quotient, and remainder.

$$\begin{array}{ccccccc}
\textbf{divisor} & \cdot & \text{quotient} & + & \text{remainder} & = & \textbf{dividend} \\
(x+2) & \cdot & (x+3) & + & (-10) & = & x^2+5x-4
\end{array}$$

This leads to a definition for polynomial division.

KeyConcept Polynomial Division

Let $f(x)$ and $d(x)$ be polynomials such that the degree of $d(x)$ is less than or equal to the degree of $f(x)$ and $d(x) \neq 0$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

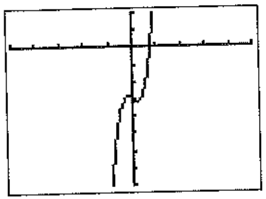
$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \quad \text{or} \quad f(x) = d(x) \cdot q(x) + r(x),$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If $r(x) = 0$, then $d(x)$ *divides evenly* into $f(x)$.

Before dividing, be sure that each polynomial is written in standard form and that placeholders with zero coefficients are inserted where needed for missing powers of the variable.

StudyTip

Graphical Check You can also check the result in Example 2 using a graphing calculator. The graphs of $Y_1 = 9x^3 - x - 3$ and $Y_2 = (3x^2 - 2x + 1) \cdot (3x + 2) - 5$ are identical.



[−5, 5] scl: 1 by [−8, 2] scl: 1

Example 2 Long Division with Nonzero Remainder

Divide $9x^3 - x - 3$ by $3x + 2$.

First rewrite $9x^3 - x - 3$ as $9x^2 + 0x^2 - x - 3$. Then divide.

$$\begin{array}{r}
3x^2-2x+1 \\
3x+2\overline{)9x^3+0x^2-x-3} \\
(-)\ \underline{9x^3+6x^2} \\
-6x^2-x \\
(-)\ \underline{-6x^2-4x} \\
3x-3 \\
(-)\ \underline{3x+2} \\
-5
\end{array}$$

You can write this result as

$$\frac{9x^3-x-3}{3x+2} = 3x^2-2x+1+\frac{-5}{3x+2}, x \neq -\frac{2}{3}$$
$$= 3x^2-2x+1-\frac{5}{3x+2}, x \neq -\frac{2}{3}.$$

CHECK Multiply to check this result.

$$(3x+2)(3x^2-2x+1)+(-5) \stackrel{?}{=} 9x^3-x-3$$
$$9x^3-6x^2+3x+6x^2-4x+2-5 \stackrel{?}{=} 9x^3-x-3$$
$$9x^3-x-3 = 9x^3-x-3 \checkmark$$

GuidedPractice

Divide using long division.

2A. $(8x^3 - 18x^2 + 21x - 20) \div (2x - 3)$

2B. $(-3x^3 + x^2 + 4x - 66) \div (x - 5)$

When dividing polynomials, the divisor can have a degree higher than 1. This can sometimes result in a quotient with missing terms.

StudyTip

Division by Zero In Example 3, this division is not defined for $x^2 - 2x + 7 = 0$. From this point forward in this lesson, you can assume that x cannot take on values for which the indicated division is undefined.

Example 3 Division by Polynomial of Degree 2 or Higher

Divide $2x^4 - 4x^3 + 13x^2 + 3x - 11$ by $x^2 - 2x + 7$.

$$\begin{array}{r} 2x^2 \qquad\qquad -1 \\ x^2 - 2x + 7\overline{)2x^4 - 4x^3 + 13x^2 + 3x - 11} \\ (-)\ \underline{2x^4 - 4x^3 + 14x^2} \qquad\qquad\quad \\ -x^2 + 3x - 11 \\ (-)\ \underline{-x^2 + 2x - 7} \\ x - 4 \end{array}$$

You can write this result as $\frac{2x^4 - 4x^3 + 13x^2 + 3x - 11}{x^2 - 2x + 7} = 2x^2 - 1 + \frac{x - 4}{x^2 - 2x + 7}$.

GuidedPractice

Divide using long division.

3A. $(2x^3 + 5x^2 - 7x + 6) \div (x^2 + 3x - 4)$

3B. $(6x^5 - x^4 + 12x^2 + 15x) \div (3x^3 - 2x^2 + x)$

Synthetic division is a shortcut for dividing a polynomial by a linear factor of the form $x - c$. Consider the long division from Example 1.

Long Division	Suppress Variables	Collapse Vertically	Synthetic Division
Notice the coefficients highlighted in colored text. $\begin{array}{r} 6x^2 - 7x - 3 \\ x - 3\overline{)6x^3 - 25x^2 + 18x + 9} \\ (-)\ \underline{6x^3 - 18x^2} \qquad\qquad\quad \\ -7x^2 + 18x \qquad \\ (-)\ \underline{-7x^2 + 21x} \qquad \\ -3x + 9 \\ (-)\ \underline{-3x + 9} \\ 0 \end{array}$	Suppress x and powers of x. $\begin{array}{r} 6 \quad -7 \quad -3 \\ -3\overline{)6 \quad -25 \quad +18 \quad +9} \\ (-)\ \underline{6 \quad -18} \qquad\qquad\quad \\ -7 \quad +18 \qquad \\ (-)\ \underline{-7 \quad +21} \qquad \\ -3 \quad +9 \\ (-)\ \underline{-3 \quad +9} \\ 0 \end{array}$	Collapse the long division vertically, eliminating duplications. $\begin{array}{r\|rrr\|r} \underline{-3} & 6 & -25 & 18 & 9 \\ & & -18 & 21 & 9 \\ \hline & 6 & -7 & -3 & 0 \end{array}$	Change the signs of the divisor and the numbers on the second line. $\begin{array}{r\|rrr\|r} \underline{3} & 6 & -25 & 18 & 9 \\ & & 18 & -21 & -9 \\ \hline & 6 & -7 & -3 & 0 \end{array}$ The number now representing the divisor is the related zero of the binomial $x - c$. Also, by changing the signs on the second line, we are now adding instead of subtracting.

We can use the synthetic division shown in the example above to outline a procedure for synthetic division of any polynomial by a binomial.

KeyConcept Synthetic Division Algorithm

To divide a polynomial by the factor $x - c$, complete each step.

Step 1 Write the coefficients of the dividend in standard form. Write the related zero c of the divisor $x - c$ in the box. Bring down the first coefficient.

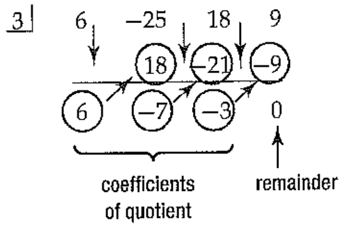
Step 2 Multiply the first coefficient by c. Write the product under the second coefficient.

Step 3 Add the product and the second coefficient.

Step 4 Repeat Steps 2 and 3 until you reach a sum in the last column. The numbers along the bottom row are the coefficients of the quotient. The power of the first term is one less than the degree of the dividend. The final number is the remainder.

Example

Divide $6x^3 - 25x^2 + 18x + 9$ by $x - 3$.



↓ = Add terms. ↗ = Multiply by c, and write the product.

As with division of polynomials by long division, remember to use zeros as placeholders for any missing terms in the dividend. When a polynomial is divided by one of its binomial factors $x - c$, the quotient is called a **depressed polynomial.**

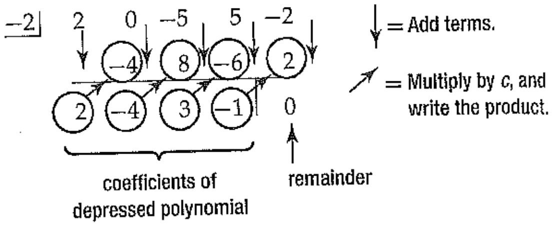
Example 4 Synthetic Division

Divide using synthetic division.

a. $(2x^4 - 5x^2 + 5x - 2) \div (x + 2)$

Because $x + 2 = x - (-2)$, $c = -2$. Set up the synthetic division as follows, using zero as a placeholder for the missing x^3-term in the dividend. Then follow the synthetic division procedure.

$$\begin{array}{r|rrrrr} -2 & 2 & 0 & -5 & 5 & -2 \\ & & & & & \\ \hline & & & & & \end{array}$$



The quotient has degree one less than that of the dividend, so

$$\frac{2x^4 - 5x^2 + 5x - 2}{x + 2} = 2x^3 - 4x^2 + 3x - 1.$$

Check this result.

TechnologyTip

Using Graphs To check your division, you can graph the polynomial division expression and the depressed polynomial with the remainder. The graphs should coincide.

b. $(10x^3 - 13x^2 + 5x - 14) \div (2x - 3)$

Rewrite the division expression so that the divisor is of the form $x - c$.

$$\frac{10x^3 - 13x^2 + 5x - 14}{2x - 3} = \frac{(10x^3 - 13x^2 + 5x - 14) \div 2}{(2x - 3) \div 2} \text{ or } \frac{5x^3 - \frac{13}{2}x^2 + \frac{5}{2}x - 7}{x - \frac{3}{2}}.$$

So, $c = \frac{3}{2}$. Perform the synthetic division.

$$\begin{array}{r|rrrr} \frac{3}{2} & 5 & -\frac{13}{2} & \frac{5}{2} & -7 \\ & & & & \\ \hline & & & & \end{array} \quad \Rightarrow \quad \begin{array}{r|rrrr} \frac{3}{2} & 5 & -\frac{13}{2} & \frac{5}{2} & -7 \\ & & \frac{15}{2} & \frac{3}{2} & 6 \\ \hline & 5 & 1 & 4 & -1 \end{array}$$

So, $\frac{10x^3 - 13x^2 + 5x - 14}{2x - 3} = 5x^2 + x + 4 - \frac{1}{x - \frac{3}{2}}$ or $5x^2 + x + 4 - \frac{2}{2x - 3}$.

Check this result.

GuidedPractice

4A. $(4x^3 + 3x^2 - x + 8) \div (x - 3)$

4B. $(6x^4 + 11x^3 - 15x^2 - 12x + 7) \div (3x + 1)$

2 The Remainder and Factor Theorems

When $d(x)$ is the divisor $(x - c)$ with degree 1, the remainder is the real number r. So, the division algorithm simplifies to

$$f(x) = (x - c) \cdot q(x) + r.$$

Evaluating $f(x)$ for $x = c$, we find that

$$f(c) = (c - c) \cdot q(c) + r = 0 \cdot q(c) + r \text{ or } r.$$

So, $f(c) = r$, which is the remainder. This leads us to the following theorem.

KeyConcept Remainder Theorem

If a polynomial $f(x)$ is divided by $x - c$, the remainder is $r = f(c)$.

The Remainder Theorem indicates that to evaluate a polynomial function $f(x)$ for $x = c$, you can divide $f(x)$ by $x - c$ using synthetic division. The remainder will be $f(c)$. Using synthetic division to evaluate a function is called **synthetic substitution.**

Real-WorldLink

High school football rules are similar to most college and professional football rules. Two major differences are that the quarters are 12 minutes as opposed to 15 minutes and kick-offs take place at the 40-yard line instead of the 30-yard line.

Source: National Federation of State High School Associations

Real-World Example 5 Use the Remainder Theorem

FOOTBALL The number of tickets sold during the Northside High School football season can be modeled by $t(x) = x^3 - 12x^2 + 48x + 74$, where x is the number of games played. Use the Remainder Theorem to find the number of tickets sold during the twelfth game of the Northside High School football season.

To find the number of tickets sold during the twelfth game, use synthetic substitution to evaluate $t(x)$ for $x = 12$.

$$\begin{array}{r|rrrr} 12 & 1 & -12 & 48 & 74 \\ & & 12 & 0 & 576 \\ \hline & 1 & 0 & 48 & \vert\ 650 \end{array}$$

The remainder is 650, so $t(12) = 650$. Therefore, 650 tickets were sold during the twelfth game of the season.

CHECK You can check your answer using direct substitution.

$t(x) = x^3 - 12x^2 + 48x + 74$ Original function

$t(12) = (12)^3 - 12(12)^2 + 48(12) + 74$ or 650 ✓ Substitute 12 for x and simplify.

Guided Practice

5. FOOTBALL Use the Remainder Theorem to determine the number of tickets sold during the thirteenth game of the season.

If you use the Remainder Theorem to evaluate $f(x)$ at $x = c$ and the result is $f(c) = 0$, then you know that c is a zero of the function and $(x - c)$ is a factor. This leads us to another useful theorem that provides a test to determine whether $(x - c)$ is a factor of $f(x)$.

KeyConcept Factor Theorem

A polynomial $f(x)$ has a factor $(x - c)$ if and only if $f(c) = 0$.

You can use synthetic division to perform this test.

Example 6 Use the Factor Theorem

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$.

a. $f(x) = 4x^4 + 21x^3 + 25x^2 - 5x + 3$; $(x - 1)$, $(x + 3)$

Use synthetic division to test each factor, $(x - 1)$ and $(x + 3)$.

$$\begin{array}{r|rrrrr} 1 & 4 & 21 & 25 & -5 & 3 \\ & & 4 & 25 & 50 & 45 \\ \hline & 4 & 25 & 50 & 45 & \vert\ 48 \end{array}$$

$$\begin{array}{r|rrrrr} -3 & 4 & 21 & 25 & -5 & 3 \\ & & -12 & -27 & 6 & -3 \\ \hline & 4 & 9 & -2 & 1 & \vert\ 0 \end{array}$$

Because the remainder when $f(x)$ is divided by $(x - 1)$ is 48, $f(1) = 48$ and $(x - 1)$ is not a factor.

Because the remainder when $f(x)$ is divided by $(x + 3)$ is 0, $f(-3) = 0$ and $(x + 3)$ is a factor.

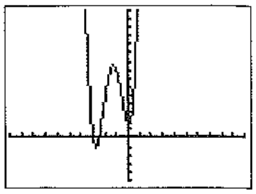
Because $(x + 3)$ is a factor of $f(x)$, we can use the quotient of $f(x) \div (x + 3)$ to write a factored form of $f(x)$.

$$f(x) = (x + 3)(4x^3 + 9x^2 - 2x + 1)$$

TechnologyTip

Zeros You can confirm the zeros on the graph of a function by using the zero feature on the CALC menu of a graphing calculator.

CHECK If $(x + 3)$ is a factor of $f(x) = 4x^4 + 21x^3 + 25x^2 - 5x + 3$, then -3 is a zero of the function and $(-3, 0)$ is an x-intercept of the graph. Graph $f(x)$ using a graphing calculator and confirm that $(-3, 0)$ is a point on the graph. ✓



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b. $f(x) = 2x^3 - x^2 - 41x - 20$; $(x + 4)$, $(x - 5)$

Use synthetic division to test the factor $(x + 4)$.

$$\begin{array}{r|rrrr} -4 & 2 & -1 & -41 & -20 \\ & & -8 & 36 & 20 \\ \hline & 2 & -9 & -5 & \big| \; 0 \end{array}$$

Because the remainder when $f(x)$ is divided by $(x + 4)$ is 0, $f(-4) = 0$ and $(x + 4)$ is a factor of $f(x)$.

Next, test the second factor, $(x - 5)$, with the depressed polynomial $2x^2 - 9x - 5$.

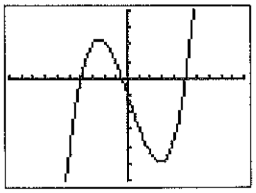
$$\begin{array}{r|rrr} 5 & 2 & -9 & -5 \\ & & 10 & 5 \\ \hline & 2 & 1 & \big| \; 0 \end{array}$$

Because the remainder when the quotient of $f(x) \div (x + 4)$ is divided by $(x - 5)$ is 0, $f(5) = 0$ and $(x - 5)$ is a factor of $f(x)$.

Because $(x + 4)$ and $(x - 5)$ are factors of $f(x)$, we can use the final quotient to write a factored form of $f(x)$.

$$f(x) = (x + 4)(x - 5)(2x + 1)$$

CHECK The graph of $f(x) = 2x^3 - x^2 - 41x - 20$ confirms that $x = -4$, $x = 5$, and $x = -\frac{1}{2}$ are zeros of the function. ✓



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GuidedPractice

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$.

6A. $f(x) = 3x^3 - x^2 - 22x + 24$; $(x - 2)$, $(x + 5)$

6B. $f(x) = 4x^3 - 34x^2 + 54x + 36$; $(x - 6)$, $(x - 3)$

You can see that synthetic division is a useful tool for factoring and finding the zeros of polynomial functions.

ConceptSummary Synthetic Division and Remainders

If r is the remainder obtained after a synthetic division of $f(x)$ by $(x - c)$, then the following statements are true.

- r is the value of $f(c)$.
- If $r = 0$, then $(x - c)$ is a factor of $f(x)$.
- If $r = 0$, then c is an x-intercept of the graph of f.
- If $r = 0$, then $x = c$ is a solution of $f(x) = 0$.

Exercises

○ = Step-by-Step Solutions begin on page R29.

Factor each polynomial completely using the given factor and long division. (Example 1)

1. $x^3 + 2x^2 - 23x - 60; x + 4$
2. $x^3 + 2x^2 - 21x + 18; x - 3$
3. $x^3 + 3x^2 - 18x - 40; x - 4$
4. $4x^3 + 20x^2 - 8x - 96; x + 3$
5. $-3x^3 + 15x^2 + 108x - 540; x - 6$
6. $6x^3 - 7x^2 - 29x - 12; 3x + 4$
7. $x^4 + 12x^3 + 38x^2 + 12x - 63; x^2 + 6x + 9$
8. $x^4 - 3x^3 - 36x^2 + 68x + 240; x^2 - 4x - 12$

Divide using long division. (Examples 2 and 3)

9. $(5x^4 - 3x^3 + 6x^2 - x + 12) \div (x - 4)$
10. $(x^6 - 2x^5 + x^4 - x^3 + 3x^2 - x + 24) \div (x + 2)$
11. $(4x^4 - 8x^3 + 12x^2 - 6x + 12) \div (2x + 4)$
12. $(2x^4 - 7x^3 - 38x^2 + 103x + 60) \div (x - 3)$
13. $(6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6) \div (2x - 1)$
14. $(108x^5 - 36x^4 + 75x^2 + 36x + 24) \div (3x + 2)$
15. $(x^4 + x^3 + 6x^2 + 18x - 216) \div (x^3 - 3x^2 + 18x - 54)$
16. $(4x^4 - 14x^3 - 14x^2 + 110x - 84) \div (2x^2 + x - 12)$
17. $\frac{6x^5 - 12x^4 + 10x^3 - 2x^2 - 8x + 8}{3x^3 + 2x + 3}$
18. $\frac{12x^5 + 5x^4 - 15x^3 + 19x^2 - 4x - 28}{3x^3 + 2x^2 - x + 6}$

Divide using synthetic division. (Example 4)

19. $(x^4 - x^3 + 3x^2 - 6x - 6) \div (x - 2)$
20. $(2x^4 + 4x^3 - 2x^2 + 8x - 4) \div (x + 3)$

㉑ $(3x^4 - 9x^3 - 24x - 48) \div (x - 4)$

22. $(x^5 - 3x^3 + 6x^2 + 9x + 6) \div (x + 2)$
23. $(12x^5 + 10x^4 - 18x^3 - 12x^2 - 8) \div (2x - 3)$
24. $(36x^4 - 6x^3 + 12x^2 - 30x - 12) \div (3x + 1)$
25. $(45x^5 + 6x^4 + 3x^3 + 8x + 12) \div (3x - 2)$
26. $(48x^5 + 28x^4 + 68x^3 + 11x + 6) \div (4x + 1)$
27. $(60x^6 + 78x^5 + 9x^4 - 12x^3 - 25x - 20) \div (5x + 4)$
28. $\frac{16x^6 - 56x^5 - 24x^4 + 96x^3 - 42x^2 - 30x + 105}{2x - 7}$

29. EDUCATION The number of U.S. students, in thousands, that graduated with a bachelor's degree from 1970 to 2006 can be modeled by $g(x) = 0.0002x^5 - 0.016x^4 + 0.512x^3 - 7.15x^2 + 47.52x + 800.27$, where x is the number of years since 1970. Use synthetic substitution to find the number of students that graduated in 2005. Round to the nearest thousand. (Example 5)

30. SKIING The distance in meters that a person travels on skis can be modeled by $d(t) = 0.2t^2 + 3t$, where t is the time in seconds. Use the Remainder Theorem to find the distance traveled after 45 seconds. (Example 5)

Find each $f(c)$ using synthetic substitution. (Example 5)

31. $f(x) = 4x^5 - 3x^4 + x^3 - 6x^2 + 8x - 15; c = 3$

32. $f(x) = 3x^6 - 2x^5 + 4x^4 - 2x^3 + 8x - 3; c = 4$

33. $f(x) = 2x^6 + 5x^5 - 3x^4 + 6x^3 - 9x^2 + 3x - 4; c = 5$

34. $f(x) = 4x^6 + 8x^5 - 6x^3 - 5x^2 + 6x - 4; c = 6$

35. $f(x) = 10x^5 + 6x^4 - 8x^3 + 7x^2 - 3x + 8; c = -6$

36. $f(x) = -6x^7 + 4x^5 - 8x^4 + 12x^3 - 15x^2 - 9x + 64; c = 2$

37. $f(x) = -2x^8 + 6x^5 - 4x^4 + 12x^3 - 6x + 24; c = 4$

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$. (Example 6)

38. $f(x) = x^4 - 2x^3 - 9x^2 + x + 6; (x + 2), (x - 1)$

39. $f(x) = x^4 + 2x^3 - 5x^2 + 8x + 12; (x - 1), (x + 3)$

40. $f(x) = x^4 - 2x^3 + 24x^2 + 18x + 135; (x - 5), (x + 5)$

41. $f(x) = 3x^4 - 22x^3 + 13x^2 + 118x - 40; (3x - 1), (x - 5)$

42. $f(x) = 4x^4 - x^3 - 36x^2 - 111x + 30; (4x - 1), (x - 6)$

43. $f(x) = 3x^4 - 35x^3 + 38x^2 + 56x + 64; (3x - 2), (x + 2)$

44. $f(x) = 5x^5 + 38x^4 - 68x^2 + 59x + 30; (5x - 2), (x + 8)$

45. $f(x) = 4x^5 - 9x^4 + 39x^3 + 24x^2 + 75x + 63; (4x + 3), (x - 1)$

46. TREES The height of a tree in feet at various ages in years is given in the table.

Age	Height	Age	Height
2	3.3	24	73.8
6	13.8	26	82.0
10	23.0	28	91.9
14	42.7	30	101.7
20	60.7	36	111.5

a. Use a graphing calculator to write a quadratic equation to model the growth of the tree.

b. Use synthetic division to evaluate the height of the tree at 15 years.

47. BICYCLING Patrick is cycling at an initial speed v_0 of 4 meters per second. When he rides downhill, the bike accelerates at a rate a of 0.4 meter per second squared. The vertical distance from the top of the hill to the bottom of the hill is 25 meters. Use $d(t) = v_0t + \frac{1}{2}at^2$ to find how long it will take Patrick to ride down the hill, where $d(t)$ is distance traveled and t is given in seconds.

Factor each polynomial using the given factor and long division. Assume $n > 0$.

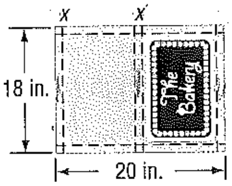
48. $x^{3n} + x^{2n} - 14x^n - 24;\ x^n + 2$

49. $x^{3n} + x^{2n} - 12x^n + 10;\ x^n - 1$

50. $4x^{3n} + 2x^{2n} - 10x^n + 4;\ 2x^n + 4$

51. $9x^{3n} + 24x^{2n} - 171x^n + 54;\ 3x^n - 1$

52. MANUFACTURING An 18-inch by 20-inch sheet of cardboard is cut and folded into a bakery box.



- **a.** Write a polynomial function to model the volume of the box.
- **b.** Graph the function.
- **c.** The company wants the box to have a volume of 196 cubic inches. Write an equation to model this situation.
- **d.** Find a positive integer for x that satisfies the equation found in part **c.**

Find the value of k so that each remainder is zero.

53. $\dfrac{x^3 - kx^2 + 2x - 4}{x - 2}$

54. $\dfrac{x^3 + 18x^2 + kx + 4}{x + 2}$

55. $\dfrac{x^3 + 4x^2 - kx + 1}{x + 1}$

56. $\dfrac{2x^3 - x^2 + x + k}{x - 1}$

57. SCULPTING Esteban will use a block of clay that is 3 feet by 4 feet by 5 feet to make a sculpture. He wants to reduce the volume of the clay by removing the same amount from the length, the width, and the height.

- **a.** Write a polynomial function to model the situation.
- **b.** Graph the function.
- **c.** He wants to reduce the volume of the clay to $\frac{3}{5}$ of the original volume. Write an equation to model the situation.
- **d.** How much should he take from each dimension?

Use the graphs and synthetic division to completely factor each polynomial.

58. $f(x) = 8x^4 + 26x^3 - 103x^2 - 156x + 45$ (Figure 2.3.1)

59. $f(x) = 6x^5 + 13x^4 - 153x^3 + 54x^2 + 724x - 840$ (Figure 2.3.2)

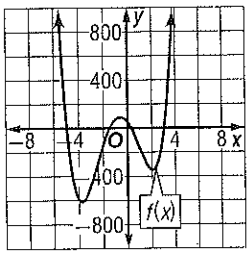


Figure 2.3.1

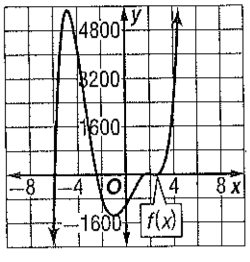


Figure 2.3.2

60. MULTIPLE REPRESENTATIONS In this problem, you will explore the upper and lower bounds of a function.

- **a. GRAPHICAL** Graph each related polynomial function, and determine the greatest and least zeros. Then copy and complete the table.

Polynomial	Greatest Zero	Least Zero
$x^3 - 2x^2 - 11x + 12$		
$x^4 + 6x^3 + 3x^2 - 10x$		
$x^5 - x^4 - 2x^3$		

- **b. NUMERICAL** Use synthetic division to evaluate each function in part **a** for three integer values greater than the greatest zero.
- **c. VERBAL** Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for an integer greater than its greatest zero.
- **d. NUMERICAL** Use synthetic division to evaluate each function in part **a** for three integer values less than the least zero.
- **e. VERBAL** Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for a number less than its least zero.

H.O.T. Problems Use Higher-Order Thinking Skills

(61) CHALLENGE Is $(x - 1)$ a factor of $18x^{165} - 15x^{135} + 8x^{105} - 15x^{55} + 4$? Explain your reasoning.

62. WRITING IN MATH Explain how you can use a graphing calculator, synthetic division, and factoring to completely factor a fifth-degree polynomial with rational coefficients, three integral zeros, and two non-integral, rational zeros.

63. REASONING Determine whether the statement below is *true* or *false*. Explain.

If $h(y) = (y + 2)(3y^2 + 11y - 4) - 1$, then the remainder of $\frac{h(y)}{y + 2}$ is -1.

CHALLENGE **Find k so that the quotient has a 0 remainder.**

64. $\dfrac{x^3 + kx^2 - 34x + 56}{x + 7}$

65. $\dfrac{x^6 + kx^4 - 8x^3 + 173x^2 - 16x - 120}{x - 1}$

66. $\dfrac{kx^3 + 2x^2 - 22x - 4}{x - 2}$

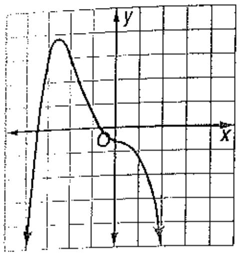
67. CHALLENGE If $2x^2 - dx + (31 - d^2)x + 5$ has a factor $x - d$, what is the value of d if d is an integer?

68. WRITING IN MATH Compare and contrast polynomial division using long division and using synthetic division.

Spiral Review

Determine whether the degree n of the polynomial for each graph is *even* or *odd* and whether its leading coefficient a_n is *positive* or *negative*. (Lesson 2-2)

69.



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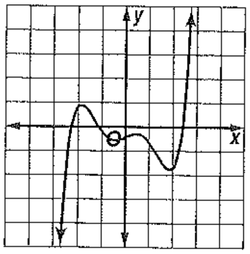

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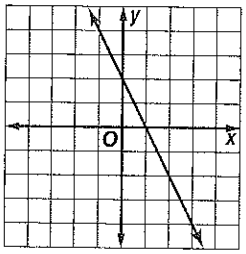


72. SKYDIVING The approximate time t in seconds that it takes an object to fall a distance of d feet is given by $t = \sqrt{\frac{d}{16}}$. Suppose a skydiver falls 11 seconds before the parachute opens. How far does the skydiver fall during this time period? (Lesson 2–1)

73. FIRE FIGHTING The velocity v and maximum height h of water being pumped into the air are related by $v = \sqrt{2gh}$, where g is the acceleration due to gravity (32 feet/second^2). (Lesson 1–7)

a. Determine an equation that will give the maximum height of the water as a function of its velocity.

b. The Mayfield Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 feet/second meet the fire department's needs? Explain.

Solve each system of equations algebraically. (Lesson 0–5)

74. $5x - y = 16$
$2x + 3y = 3$

75. $3x - 5y = -8$
$x + 2y = 1$

76. $y = 6 - x$
$x = 4.5 + y$

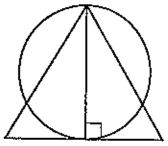
77. $2x + 5y = 4$
$3x + 6y = 5$

78. $7x + 12y = 16$
$5y - 4x = -21$

79. $4x + 5y = -8$
$3x - 7y = 10$

Skills Review for Standardized Tests

80. SAT/ACT In the figure, an equilateral triangle is drawn with an altitude that is also the diameter of the circle. If the perimeter of the triangle is 36, what is the circumference of the circle?



A $6\sqrt{2}\pi$
B $6\sqrt{3}\pi$
C $12\sqrt{2}\pi$
D $12\sqrt{3}\pi$
E 36π

81. REVIEW If (3, −7) is the center of a circle and (8, 5) is on the circle, what is the circumference of the circle?

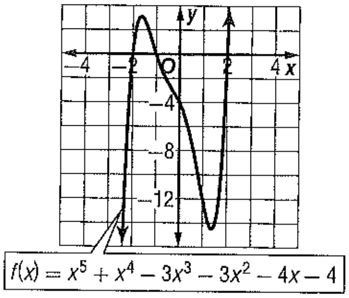
F 13π
G 15π
H 18π
J 25π
K 26π

82. REVIEW The first term in a sequence is x. Each subsequent term is three less than twice the preceding term. What is the 5th term in the sequence?

A $8x - 21$
B $8x - 15$
C $16x - 39$
D $16x - 45$
E $32x - 43$

83. Use the graph of the polynomial function. Which is not a factor of $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$?

F $(x - 2)$
G $(x + 2)$
H $(x - 1)$
J $(x + 1)$



CHAPTER 2

Mid-Chapter Quiz

Lessons 2-1 through 2-3

Graph and analyze each function. Describe its domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Lesson 2-1)

1. $f(x) = 2x^3$

2. $f(x) = -\frac{2}{3}x^4$

3. $f(x) = 3x^{-8}$

4. $f(x) = 4x^{\frac{2}{5}}$

5. TREES The heights of several fir trees and the areas under their branches are shown in the table. (Lesson 2-1)

Height (m)	Area (m²)
4.2	37.95
2.1	7.44
3.4	23.54
1.7	4.75
4.6	46.48

a. Create a scatter plot of the data.

b. Determine a power function to model the data.

c. Predict the area under the branches of a fir tree that is 7.6 meters high.

Solve each equation. (Lesson 2-1)

6. $\sqrt{5x+7} = 13$

7. $\sqrt{2x-2} + 1 = x$

8. $\sqrt{3x+10} + 1 = \sqrt{x+11}$

9. $-5 = \sqrt[4]{(6x+3)^3} - 32$

State the number of possible real zeros and turning points of each function. Then find all of the real zeros by factoring. (Lesson 2-2)

10. $f(x) = x^2 - 11x - 26$

11. $f(x) = 3x^5 + 2x^4 - x^3$

12. $f(x) = x^4 + 9x^2 - 10$

13. MULTIPLE CHOICE Which of the following describes the possible end behavior of a polynomial of odd degree? (Lesson 2-2)

A $\lim_{x \to \infty} f(x) = 5$; $\lim_{x \to -\infty} f(x) = 5$

B $\lim_{x \to \infty} f(x) = -\infty$; $\lim_{x \to -\infty} f(x) = -\infty$

C $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to -\infty} f(x) = \infty$

D $\lim_{x \to \infty} f(x) = -\infty$; $\lim_{x \to -\infty} f(x) = \infty$

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test. (Lesson 2-2)

14. $f(x) = -7x^4 - 3x^3 - 8x^2 + 23x + 7$

15. $f(x) = -5x^5 + 4x^4 + 12x^2 - 8$

16. ENERGY Crystal's electricity consumption measured in kilowatt hours (kWh) for the past 12 months is shown below. (Lesson 2-2)

Month	Consumption (kWh)	Month	Consumption (kWh)
January	240	July	300
February	135	August	335
March	98	September	390
April	110	October	345
May	160	November	230
June	230	December	100

a. Determine a model for the number of kilowatt hours Crystal used as a function of the number of months since January.

b. Use the model to predict how many kilowatt hours Crystal will use the following January. Does this answer make sense? Explain your reasoning.

Divide using synthetic division. (Lesson 2-3)

17. $(5x^3 - 7x^2 + 8x - 13) \div (x - 1)$

18. $(x^4 - x^3 - 9x + 18) \div (x - 2)$

19. $(2x^3 - 11x^2 + 9x - 6) \div (2x - 1)$

Determine each $f(c)$ using synthetic substitution. (Lesson 2-3)

20. $f(x) = 9x^5 + 4x^4 - 3x^3 + 18x^2 - 16x + 8$; $c = 2$

21. $f(x) = 6x^6 - 3x^5 + 8x^4 + 12x^2 - 6x + 4$; $c = -3$

22. $f(x) = -2x^6 + 8x^5 - 12x^4 + 9x^3 - 8x^2 + 6x - 3$; $c = -2$

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$. (Lesson 2-3)

23. $f(x) = x^3 + 2x^2 - 25x - 50$; $(x + 5)$

24. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$; $(x - 1)$, $(x - 2)$

25. MULTIPLE CHOICE Find the remainder when $f(x) = x^3 - 4x + 5$ is divided by $x + 3$. (Lesson 2-3)

F -10

G 8

H 20

J 26

LESSON 2-4 Zeros of Polynomial Functions

Then

- You learned that a polynomial function of degree n can have at most n real zeros. (Lesson 2-1)

Now

1. Find real zeros of polynomial functions.
2. Find complex zeros of polynomial functions.

Why?

- A company estimates that the profit P in thousands of dollars from a certain model of video game controller is given by $P(x) = -0.0007x^2 + 2.45x$, where x is the number of thousands of dollars spent marketing the controller. To find the number of advertising dollars the company should spend to make a profit of \$1,500,000, you can use techniques presented in this lesson to solve the polynomial equation $P(x) = 1500$.

NewVocabulary

Rational Zero Theorem
lower bound
upper bound
Descartes' Rule of Signs
Fundamental Theorem of Algebra
Linear Factorization Theorem
Conjugate Root Theorem
complex conjugates
irreducible over the reals

1 Real Zeros

Recall that a polynomial function of degree n can have at most n real zeros. These real zeros are either rational or irrational.

Rational Zeros	Irrational Zeros
$f(x) = 3x^2 + 7x - 6$ or $f(x) = (x + 3)(3x - 2)$ There are two rational zeros, -3 or $\frac{2}{3}$.	$g(x) = x^2 - 5$ or $g(x) = (x + \sqrt{5})(x - \sqrt{5})$ There are two irrational zeros, $\pm\sqrt{5}$.

The **Rational Zero Theorem** describes how the leading coefficient and constant term of a polynomial function with integer coefficients can be used to determine a list of all possible rational zeros.

KeyConcept Rational Zero Theorem

If f is a polynomial function of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, with degree $n \geq 1$, integer coefficients, and $a_0 \neq 0$, then every rational zero of f has the form $\frac{p}{q}$, where

- p and q have no common factors other than ±1,
- p is an integer factor of the constant term a_0, and
- q is an integer factor of the leading coefficient a_n.

Corollary If the leading coefficient a_n is 1, then any rational zeros of f are integer factors of the constant term a_0.

Once you know all of the *possible* rational zeros of a polynomial function, you can then use direct or synthetic substitution to determine which, if any, are actual zeros of the polynomial.

Example 1 Leading Coefficient Equal to 1

List all possible rational zeros of each function. Then determine which, if any, are zeros.

a. $f(x) = x^3 + 2x + 1$

Step 1 Identify possible rational zeros.

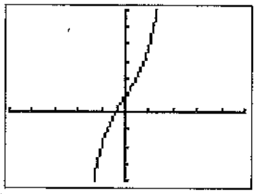
Because the leading coefficient is 1, the possible rational zeros are the integer factors of the constant term 1. Therefore, the possible rational zeros of f are 1 and -1.

Step 2 Use direct substitution to test each possible zero.

$f(1) = (1)^3 + 2(1) + 1$ or 4

$f(-1) = (-1)^3 + 2(-1) + 1$ or -2

Because $f(1) \neq 0$ and $f(-1) \neq 0$, you can conclude that f has no rational zeros. From the graph of f you can see that f has one real zero. Applying the Rational Zeros Theorem shows that this zero is irrational.



[−5, 5] scl: 1 by [−4, 6] scl: 1

b. $g(x) = x^4 + 4x^3 - 12x - 9$

Step 1 Because the leading coefficient is 1, the possible rational zeros are the integer factors of the constant term -9. Therefore, the possible rational zeros of g are ±1, ±3, and ±9.

Step 2 Begin by testing 1 and -1 using synthetic substitution.

$$\begin{array}{r|rrrr|r} 1 & 1 & 4 & 0 & -12 & -9 \\ & & 1 & 5 & 5 & -7 \\ \hline & 1 & 5 & 5 & -7 & -16 \end{array} \qquad \begin{array}{r|rrrr|r} -1 & 1 & 4 & 0 & -12 & -9 \\ & & -1 & -3 & 3 & 9 \\ \hline & 1 & 3 & -3 & -9 & 0 \end{array}$$

Because $g(-1) = 0$, you can conclude that -1 is a zero of g. Testing -3 on the depressed polynomial shows that -3 is another rational zero.

$$\begin{array}{r|rrr|r} -3 & 1 & 3 & -3 & -9 \\ & & -3 & 0 & 9 \\ \hline & 1 & 0 & -3 & 0 \end{array}$$

Thus, $g(x) = (x + 1)(x + 3)(x^2 - 3)$. Because the factor $(x^2 - 3)$ yields no rational zeros, we can conclude that g has only two rational zeros, -1 and -3.

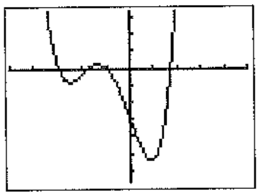

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Figure 2.4.1

CHECK The graph of $g(x) = x^4 + 4x^3 - 12x - 9$ in Figure 2.4.1 has x-intercepts at -1 and -3, and close to (2, 0) and (−2, 0). By the Rational Zeros Theorem, we know that these last two zeros must be irrational. In fact, the factor $(x^2 - 3)$ yields two irrational zeros, $\sqrt{3}$ and $-\sqrt{3}$. ✓

GuidedPractice

List all possible rational zeros of each function. Then determine which, if any, are zeros.

1A. $f(x) = x^3 + 5x^2 - 4x - 2$

1B. $h(x) = x^4 + 3x^3 - 7x^2 + 9x - 30$

When the leading coefficient of a polynomial function is not 1, the list of possible rational zeros can increase significantly.

Example 2 Leading Coefficient not Equal to 1

List all possible rational zeros of $h(x) = 3x^3 - 7x^2 - 22x + 8$. Then determine which, if any, are zeros.

Step 1 The leading coefficient is 3 and the constant term is 8.

Possible rational zeros: $\dfrac{\text{Factors of 8}}{\text{Factors of 3}} = \dfrac{\pm1, \pm2, \pm4, \pm8}{\pm1, \pm3}$ or $\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$

Step 2 By synthetic substitution, you can determine that -2 is a rational zero.

$$\begin{array}{r|rrr|r} -2 & 3 & -7 & -22 & 8 \\ & & -6 & 26 & -8 \\ \hline & 3 & -13 & 4 & 0 \end{array}$$

By the division algorithm, $h(x) = (x + 2)(3x^2 - 13x + 4)$. Once $3x^2 - 13x + 4$ is factored, the polynomial becomes $h(x) = (x + 2)(3x - 1)(x - 4)$, and you can conclude that the rational zeros of h are -2, $\frac{1}{3}$, and 4. Check this result by graphing.

GuidedPractice

List all possible rational zeros of each function. Then determine which, if any, are zeros.

2A. $g(x) = 2x^3 - 4x^2 + 18x - 36$

2B. $f(x) = 3x^4 - 18x^3 + 2x - 21$

eal-WorldLink

recent study showed that lmost a third of frequent video ame players are between 6 and 7 years old.

ource: NPD Group Inc

Real-World Example 3 Solve a Polynomial Equation

BUSINESS After the first half-hour, the number of video games that were sold by a company on their release date can be modeled by $g(x) = 2x^3 + 4x^2 - 2x$, where $g(x)$ is the number of games sold in hundreds and x is the number of hours after the release. How long did it take to sell 400 games?

Because $g(x)$ represents the number of games sold in hundreds, you need to solve $g(x) = 4$ to determine how long it will take to sell 400 games.

$$\begin{aligned} g(x) &= 4 && \text{Write the equation.} \\ 2x^3 + 4x^2 - 2x &= 4 && \text{Substitute } 2x^3 + 4x^2 - 2x \text{ for } g(x) \\ 2x^3 + 4x^2 - 2x - 4 &= 0 && \text{Subtract 4 from each side} \end{aligned}$$

Apply the Rational Zeros Theorem to this new polynomial function, $f(x) = 2x^3 + 4x^2 - 2x - 4$.

Step 1 Possible rational zeros: $\frac{\text{Factors of 4}}{\text{Factors of 2}} = \frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 2}$

$$= \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}$$

Step 2 By synthetic substitution, you can determine that 1 is a rational zero.

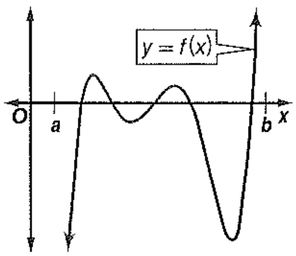
$$\begin{array}{r|rrrr} 1 & 2 & 4 & -2 & -4 \\ & & 2 & 6 & 4 \\ \hline & 2 & 6 & 4 & \multicolumn{1}{|r}{0} \end{array}$$

Because 1 is a zero of f, $x = 1$ is a solution of $f(x) = 0$. The depressed polynomial $2x^2 + 6x + 4$ can be written as $2(x + 2)(x + 1)$. The zeros of this polynomial are -2 and -1. Because time cannot be negative, the solution is $x = 1$. So, it took 1 hour to sell 400 games.

GuidedPractice

3. VOLLEYBALL A volleyball that is returned after a serve with an initial speed of 40 feet per second at a height of 4 feet is given by $f(t) = 4 + 40t - 16t^2$, where $f(t)$ is the height the ball reaches in feet and t is time in seconds. At what time(s) will the ball reach a height of 20 feet?

One way to narrow the search for real zeros is to determine an interval within which all real zeros of a function are located. A real number a is a **lower bound** for the real zeros of f if $f(x) \neq 0$ for $x < a$. Similarly, b is an **upper bound** for the real zeros of f if $f(x) \neq 0$ for $x > b$.



The real zeros of f are in the interval $[a, b]$.

You can test whether a given interval contains all real zeros of a function by using the following upper and lower bound tests.

ReadingMath

onnegative and Nonpositive emember that a nonnegative alue is one that is either ositive or zero, and a onpositive value is one that is ither negative or zero.

KeyConcept Upper and Lower Bound Tests

Let f be a polynomial function of degree $n \geq 1$, real coefficients, and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$ using synthetic division.

- If $c \leq 0$ and every number in the last line of the division is alternately nonnegative and nonpositive, then c is a *lower bound* for the real zeros of f.
- If $c \geq 0$ and every number in the last line of the division is nonnegative, then c is an *upper bound* for the real zeros of f.

To make use of the upper and lower bound tests, follow these steps.

Step 1 Graph the function to determine an interval in which the zeros lie.

Step 2 Using synthetic substitution, confirm that the upper and lower bounds of your interval are in fact upper and lower bounds of the function by applying the upper and lower bound tests.

Step 3 Use the Rational Zero Theorem to help find all the real zeros.

StudyTip

Upper and Lower Bounds Upper and lower bounds of a function are not necessarily unique.

Example 4 Use the Upper and Lower Bound Tests

Determine an interval in which all real zeros of $h(x) = 2x^4 - 11x^3 + 2x^2 - 44x - 24$ must lie. Explain your reasoning using the upper and lower bound tests. Then find all the real zeros.

Step 1 Graph $h(x)$ using a graphing calculator. From this graph, it appears that the real zeros of this function lie in the interval $[-1, 7]$.

Step 2 Test a lower bound of $c = -1$ and an upper bound of $c = 7$.

$$\begin{array}{r|rrrr|r}
-1 & 2 & -11 & 2 & -44 & -24 \\
 & & -2 & 13 & -15 & 59 \\
\hline
 & 2 & -13 & 15 & -59 & 35
\end{array}$$

Values alternate signs in the last line, so −1 is a lower bound.

$$\begin{array}{r|rrrr|r}
7 & 2 & -11 & 2 & -44 & -24 \\
 & & 14 & 21 & 161 & 819 \\
\hline
 & 2 & 3 & 23 & 117 & 795
\end{array}$$

Values are all nonnegative in last line, so 7 is an upper bound.

Step 3 Use the Rational Zero Theorem.

$$\text{Possible rational zeros: } \frac{\text{Factors of 24}}{\text{Factors of 2}} = \frac{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24}{\pm1, \pm2}$$

$$= \pm1, \pm2, \pm4, \pm6, \pm8, \pm12, \pm24, \pm\frac{1}{2}, \pm\frac{3}{2}$$

Because the real zeros are in the interval $[-1, 7]$, you can narrow this list to just ±1, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, 2, 4, or 6. From the graph, it appears that only 6 and $-\frac{1}{2}$ are reasonable.

Begin by testing 6.

$$\begin{array}{r|rrrr|r}
6 & 2 & -11 & 2 & -44 & -24 \\
 & & 12 & 6 & 48 & 24 \\
\hline
 & 2 & 1 & 8 & 4 & 0
\end{array}$$

Now test $-\frac{1}{2}$ in the depressed polynomial.

$$\begin{array}{r|rrr|r}
-\frac{1}{2} & 2 & 1 & 8 & 4 \\
 & & -1 & 0 & -4 \\
\hline
 & 2 & 0 & 8 & 0
\end{array}$$

By the division algorithm, $h(x) = 2(x - 6)\left(x + \frac{1}{2}\right)(x^2 + 4)$. Notice that the factor $(x^2 + 4)$ has no real zeros associated with it because $x^2 + 4 = 0$ has no real solutions. So, f has two real solutions that are both rational, 6 and $-\frac{1}{2}$. The graph of $h(x) = 2x^4 - 11x^3 + 2x^2 - 44x - 24$ supports this conclusion.

GuidedPractice

Determine an interval in which all real zeros of the given function must lie. Explain your reasoning using the upper and lower bound tests. Then find all the real zeros.

4A. $g(x) = 6x^4 + 70x^3 - 21x^2 + 35x - 12$

4B. $f(x) = 10x^5 - 50x^4 - 3x^3 + 22x^2 - 41x + 30$

ReadingMath

Variation in Sign A *variation in sign* occurs in a polynomial written in standard form when consecutive coefficients have opposite signs.

Another way to narrow the search for real zeros is to use **Descartes' Rule of Signs**. This rule gives us information about the number of positive and negative real zeros of a polynomial function by looking at a polynomial's variations in sign.

KeyConcept Descartes' Rule of Signs

If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is a polynomial function with real coefficients, then

- the number of *positive* real zeros of f is equal to the number of variations in sign of $f(x)$ or less than that number by some even number and
- the number of *negative* real zeros of f is the same as the number of variations in sign of $f(-x)$ or less than that number by some even number.

Example 5 Use Descartes' Rule of Signs

Describe the possible real zeros of $g(x) = -3x^3 + 2x^2 - x - 1$.

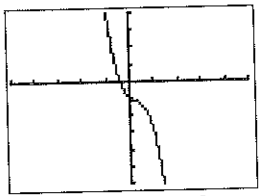
Examine the variations in sign for $g(x)$ and for $g(-x)$.

$$g(x) = \underbrace{-3x^3 +}_{-\text{ to }+} \overbrace{2x^2 -}^{+\text{ to }-} x - 1$$

$$\begin{aligned} g(-x) &= -3(-x)^3 + 2(-x)^2 - (-x) - 1 \\ &= 3x^3 + 2x^2 \underbrace{+ x -}_{+\text{ to }-} 1 \end{aligned}$$

The original function $g(x)$ has *two* variations in sign, while $g(-x)$ has *one* variation in sign. By Descartes' Rule of Signs, you know that $g(x)$ has either 2 or 0 positive real zeros and 1 negative real zero.

From the graph of $g(x)$ shown, you can see that the function has one negative real zero close to $x = -0.5$ and no positive real zeros.



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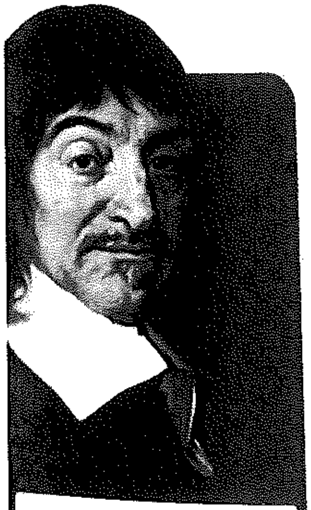
GuidedPractice

Describe the possible real zeros of each function.

5A. $h(x) = 6x^5 + 8x^2 - 10x - 15$

5B. $f(x) = -11x^4 + 20x^3 + 3x^2 - x + 18$

When using Descartes' Rule of Signs, the number of real zeros indicated includes any repeated zeros. Therefore, a zero with multiplicity m should be counted as m zeros.



Math HistoryLink

René Descartes
(1596–1650)
A French mathematician, scientist, and philosopher, Descartes wrote many philosophical works such as *Discourse on Method* and mathematical works such as *Geometry*.

2 Complex Zeros

Just as quadratic functions can have real or imaginary zeros, polynomials of higher degree can also have zeros in the complex number system. This fact, combined with the **Fundamental Theorem of Algebra**, allows us to improve our statement about the number of zeros for any nth-degree polynomial.

KeyConcept Fundamental Theorem of Algebra

A polynomial function of degree n, where $n > 0$, has at least one zero (real or imaginary) in the complex number system.

Corollary A polynomial function of degree n has *exactly* n zeros, including repeated zeros, in the complex number system.

By extending the Factor Theorem to include both real and imaginary zeros and applying the Fundamental Theorem of Algebra, we obtain the **Linear Factorization Theorem.**

KeyConcept Linear Factorization Theorem

If $f(x)$ is a polynomial function of degree $n > 0$, then f has exactly n linear factors and

$$f(x) = a_n(x - c_1)(x - c_2) \dots (x - c_n)$$

where a_n is some nonzero real number and $c_1, c_2, \dots, c_n$ are the complex zeros (including repeated zeros) of f.

According to the **Conjugate Root Theorem,** when a polynomial equation in one variable with rea coefficients has a root of the form $a + bi$, where $b \neq 0$, then its **complex conjugate,** $a - bi$, is also a root. You can use this theorem to write a polynomial function given its complex zeros.

Example 6 Find a Polynomial Function Given Its Zeros

Write a polynomial function of least degree with real coefficients in standard form that has -2, 4, and $3 - i$ as zeros.

Because $3 - i$ is a zero and the polynomial is to have real coefficients, you know that $3 + i$ mu also be a zero. Using the Linear Factorization Theorem and the zeros $-2, 4, 3 - i$, and $3 + i$, yo can write $f(x)$ as follows.

$f(x) = a[x - (-2)](x - 4)[x - (3 - i)][x - (3 + i)]$

StudyTip

Infinite Polynomials Because a can be any nonzero real number, there are an infinite number of polynomial functions that can be written for a given set of zeros.

While a can be any nonzero real number, it is simplest to let $a = 1$. Then write the function in standard form.

$f(x) = (1)(x + 2)(x - 4)[x - (3 - i)][x - (3 + i)]$ Let $a = 1$.

$= (x^2 - 2x - 8)(x^2 - 6x + 10)$ Multiply.

$= x^4 - 8x^3 + 14x^2 + 28x - 80$ Multiply.

Therefore, a function of least degree that has $-2, 4, 3 - i$, and $3 + i$ as zeros is $f(x) = x^4 - 8x^3$ $14x^2 + 28x - 80$ or any nonzero multiple of $f(x)$.

GuidedPractice

Write a polynomial function of least degree with real coefficients in standard form with tl given zeros.

6A. $-3, 1$ (multiplicity: 2), $4i$

6B. $2\sqrt{3}, -2\sqrt{3}, 1 + i$

In Example 6, you wrote a function with real and complex zeros. A function has complex zeros when its factored form contains a quadratic factor which is irreducible over the reals. A quadra expression is **irreducible over the reals** when it has real coefficients but no real zeros associate with it. This example illustrates the following theorem.

StudyTip

Prime Polynomials Note the difference between expressions which are irreducible over the reals and expressions which are *prime*. The expression $x^2 - 8$ is prime because it cannot be factored into expressions with integral coefficients. However, $x^2 - 8$ is *not* irreducible over the reals because there are real zeros associated with it, $\sqrt{8}$ and $-\sqrt{8}$.

KeyConcept Factoring Polynomial Functions Over the Reals

Every polynomial function of degree $n > 0$ with real coefficients can be written as the product of linear factors and irreducible quadratic factors, each with real coefficients.

As indicated by the Linear Factorization Theorem, when factoring a polynomial function ove complex number system, we can write the function as the product of only linear factors.

dyTip

g Multiplicity Sometimes a
nal zero will be a repeated
of a function. Use the graph
e function to determine
ther a rational zero should be
d using synthetic substitution
ccession.

Example 7 Factor and Find the Zeros of a Polynomial Function

Consider $k(x) = x^5 - 18x^3 + 30x^2 - 19x + 30$.

a. Write $k(x)$ as the product of linear and irreducible quadratic factors.

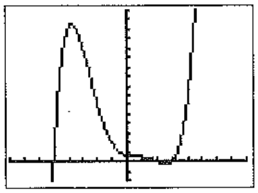
The possible rational zeros are ±1, ±2, ±3, ±5, ±6, ±10, ±15, ±30. The original polynomial has 4 sign variations.

$$k(-x) = (-x)^5 - 18(-x)^3 + 30(-x)^2 - 19(-x) + 30$$
$$= -x^5 + 18x^3 + 30x^2 + 19x + 30$$

$k(-x)$ has 1 sign variation, so $k(x)$ has 4, 2, or 0 positive real zeros and 1 negative real zero.

The graph shown suggests −5 as one real zero of $k(x)$. Use synthetic substitution to test this possibility.

$$\begin{array}{r|rrrrrr} -5 & 1 & 0 & -18 & 30 & -19 & 30 \\ & & -5 & 25 & -35 & 25 & -30 \\ \hline & 1 & -5 & 7 & -5 & 6 & 0 \end{array}$$



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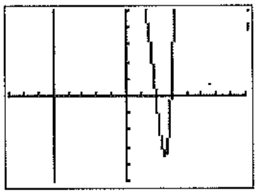
Because $k(x)$ has only 1 negative real zero, you do not need to test any other possible negative rational zeros. Zooming in on the positive real zeros in the graph suggests 2 and 3 as other rational zeros. Test these possibilities successively in the depressed quartic and then cubic polynomials.



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$$\begin{array}{r|rrrrr} 2 & 1 & -5 & 7 & -5 & 6 \\ & & 2 & -6 & 2 & -6 \\ \hline & 1 & -3 & 1 & -3 & \mathbf{0} \end{array}$$

Begin by testing 2

$$\begin{array}{r|rrrr} 3 & 1 & -3 & 1 & -3 \\ & & 3 & 0 & 3 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

Now test 3 on the depressed polynomial.

The remaining quadratic factor $(x^2 + 1)$ yields no real zeros and is therefore irreducible over the reals. So, $k(x)$ written as a product of linear and irreducible quadratic factors is $k(x) = (x + 5)(x - 2)(x - 3)(x^2 + 1)$.

dyTip

ratic Formula You could
use the Quadratic Formula to
the zeros of $x^2 + 1$ in order
ctor the expression.

$\frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}$

$\pm \frac{\sqrt{-4}}{2}$

$\pm \frac{2i}{2}$ or $\pm i$

and $-i$ are zeros and $(x - i)$
$(x + i)$ are factors.

b. Write $k(x)$ as the product of linear factors.

You can factor $x^2 + 1$ by writing the expression first as a difference of squares $x^2 - (\sqrt{-1})^2$ or $x^2 - i^2$. Then factor this difference of squares as $(x - i)(x + i)$. So, $k(x)$ written as a product of linear factors is as follows.

$$k(x) = (x + 5)(x - 2)(x - 3)(x - i)(x + i)$$

c. List all the zeros of $k(x)$.

Because the function has degree 5, by the corollary to the Fundamental Theorem of Algebra $k(x)$ has exactly five zeros, including any that may be repeated. The linear factorization gives us these five zeros: −5, 2, 3, i, and $-i$.

Guided Practice

Write each function as (a) the product of linear and irreducible quadratic factors and (b) the product of linear factors. Then (c) list all of its zeros.

7A. $f(x) = x^4 + x^3 - 26x^2 + 4x - 120$

7B. $f(x) = x^5 - 2x^4 - 2x^3 - 6x^2 - 99x + 108$

You can use synthetic substitution with complex numbers in the same way you use it with real numbers. Doing so can help you factor a polynomial in order to find all of its zeros.

WatchOut!

Complex Numbers Recall from Lesson 0-2 that all real numbers are also complex numbers.

Example 8 Find the Zeros of a Polynomial When One is Known

Find all complex zeros of $p(x) = x^4 - 6x^3 + 20x^2 - 22x - 13$ given that $2 - 3i$ is a zero of p. Then write the linear factorization of $p(x)$.

Use synthetic substitution to verify that $2 - 3i$ is a zero of $p(x)$.

$$\begin{array}{r|rrrrr} 2-3i & 1 & -6 & 20 & -22 & -13 \\ & & 2-3i & -17+6i & & \\ \hline & 1 & -4-3i & & & \end{array}$$

$(2 - 3i)(-4 - 3i) = -8 - 6i + 12i + 9i^2$
$= -8 + 6i + 9(-1)$
$= -17 + 6i$

$$\begin{array}{r|rrrrr} 2-3i & 1 & -6 & 20 & -22 & -13 \\ & & 2-3i & -17+6i & 24+3i & \\ \hline & 1 & -4-3i & 3+6i & & 0 \end{array}$$

$(2 - 3i)(3 + 6i) = 6 + 3i - 18i^2$
$= 6 + 3i - 18(-1)$
$= 24 + 3i$

$$\begin{array}{r|rrrrr} 2-3i & 1 & -6 & 20 & -22 & -13 \\ & & 2-3i & -17+6i & 24+3i & 13 \\ \hline & 1 & -4-3i & 3+6i & 2+3i & 0 \end{array}$$

$(2 - 3i)(2 + 3i) = 4 - 9i^2$
$= 4 - 9(-1)$
$= 4 + 9$ or 13

Because $2 - 3i$ is a zero of p, you know that $2 + 3i$ is also a zero of p. Divide the depressed polynomial by $2 + 3i$.

$$\begin{array}{r|rrrr} 2+3i & 1 & -4-3i & 3+6i & 2+3i \\ & & 2+3i & -4-6i & -2-3i \\ \hline & 1 & -2 & -1 & 0 \end{array}$$

Using these two zeros and the depressed polynomial from this last division, you can write $p(x) = [x - (2 - 3i)][x - (2 + 3i)](x^2 - 2x - 1)$.

Because $p(x)$ is a quartic polynomial, you know that it has exactly 4 zeros. Having found 2, you know that 2 more remain. Find the zeros of $x^2 - 2x - 1$ by using the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

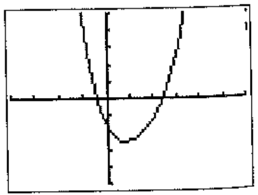
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$ $a = 1$, $b = -2$, and $c = -1$

$$= \frac{2 \pm \sqrt{8}}{2}$$ Simplify.

$$= 1 \pm \sqrt{2}$$ Simplify.

Therefore, the four zeros of $p(x)$ are $2 - 3i$, $2 + 3i$, $1 + \sqrt{2}$, and $1 - \sqrt{2}$. The linear factorization of $p(x)$ is $[x - (2 - 3i)] \cdot [x - (2 + 3i)][x - (1 + \sqrt{2})][x - (1 - \sqrt{2})]$.

Using the graph of $p(x)$, you can verify that the function has two real zeros at $1 + \sqrt{2}$ or about 2.41 and $1 - \sqrt{2}$ or about −0.41.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StudyTip

Dividing Out Common Factors Before applying any of the methods in this lesson, remember to factor out any common monomial factors. For example, $g(x) = -2x^4 + 6x^3 - 4x^2 - 8x$ should first be factored as $g(x) = -2x(x^3 - 3x^2 + 2x + 4)$, which implies that 0 is a zero of g.

Guided Practice

For each function, use the given zero to find all the complex zeros of the function. Then write the linear factorization of the function.

8A. $g(x) = x^4 - 10x^3 + 35x^2 - 46x + 10; 2 + \sqrt{3}$

8B. $h(x) = x^4 - 8x^3 + 26x^2 - 8x - 95; 1 - \sqrt{6}$

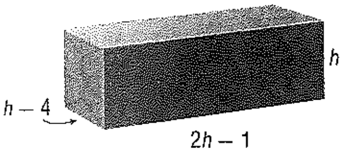
Exercises

● = Step-by-Step Solutions begin on page R29.

List all possible rational zeros of each function. Then determine which, if any, are zeros. (Examples 1 and 2)

1. $g(x) = x^4 - 6x^3 - 31x^2 + 216x - 180$
2. $f(x) = 4x^3 - 24x^2 - x + 6$
3. $g(x) = x^4 - x^3 - 31x^2 + x + 30$
4. $g(x) = -4x^4 + 35x^3 - 87x^2 + 56x + 20$
5. $h(x) = 6x^4 + 13x^3 - 67x^2 - 156x - 60$
6. $f(x) = 18x^4 + 12x^3 + 56x^2 + 48x - 64$
7. $h(x) = x^5 - 11x^4 + 49x^3 - 147x^2 + 360x - 432$
8. $g(x) = 8x^5 + 18x^4 - 5x^3 - 72x^2 - 162x + 45$

9. MANUFACTURING The specifications for the dimensions of a new cardboard container are shown. If the volume of the container is modeled by $V(h) = 2h^3 - 9h^2 + 4h$ and it will hold 45 cubic inches of merchandise, what are the container's dimensions? (Example 3)



Solve each equation. (Example 3)

10. $x^4 + 2x^3 - 7x^2 - 20x - 12 = 0$
11. $x^4 + 9x^3 + 23x^2 + 3x - 36 = 0$
12. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$
13. $x^4 - 3x^3 - 20x^2 + 84x - 80 = 0$
14. $x^4 + 34x = 6x^3 + 21x^2 - 48$
15. $6x^4 + 41x^3 + 42x^2 - 96x + 6 = -26$
16. $-12x^4 + 77x^3 = 136x^2 - 33x - 18$

17. SALES The sales $S(x)$ in thousands of dollars that a store makes during one month can be approximated by $S(x) = 2x^3 - 2x^2 + 4x$, where x is the number of days after the first day of the month. How many days will it take the store to make $16,000? (Example 3)

Determine an interval in which all real zeros of each function must lie. Explain your reasoning using the upper and lower bound tests. Then find all the real zeros. (Example 4)

18. $f(x) = x^4 - 9x^3 + 12x^2 + 44x - 48$
19. $f(x) = 2x^4 - x^3 - 29x^2 + 34x + 24$
20. $g(x) = 2x^4 + 4x^3 - 18x^2 - 4x + 16$
21. $g(x) = 6x^4 - 33x^3 - 6x^2 + 123x - 90$
22. $f(x) = 2x^4 - 17x^3 + 39x^2 - 16x - 20$
23. $f(x) = 2x^4 - 13x^3 + 21x^2 + 9x - 27$
24. $h(x) = x^5 - x^4 - 9x^3 + 5x^2 + 16x - 12$
25. $h(x) = 4x^5 - 20x^4 + 5x^3 + 80x^2 - 75x + 18$

Describe the possible real zeros of each function. (Example 5)

26. $f(x) = -2x^3 - 3x^2 + 4x + 7$
27. $f(x) = 10x^4 - 3x^3 + 8x^2 - 4x - 8$
28. $f(x) = -3x^4 - 5x^3 + 4x^2 + 2x - 6$
29. $f(x) = 12x^4 + 6x^3 + 3x^2 - 2x + 12$
30. $g(x) = 4x^5 + 3x^4 + 9x^3 - 8x^2 + 16x - 24$
31. $h(x) = -4x^5 + x^4 - 8x^3 - 24x^2 + 64x - 124$

Write a polynomial function of least degree with real coefficients in standard form that has the given zeros. (Example 6)

32. $3, -4, 6, -1$
33. $-2, -4, -3, 5$
34. $-5, 3, 4 + i$
35. $-1, 8, 6 - i$
36. $2\sqrt{5}, -2\sqrt{5}, -3, 7$
37. $-5, 2, 4 - \sqrt{3}, 4 + \sqrt{3}$
38. $\sqrt{7}, -\sqrt{7}, 4i$
39. $\sqrt{6}, -\sqrt{6}, 3 - 4i$
40. $2 + \sqrt{3}, 2 - \sqrt{3}, 4 + 5i$
41. $6 - \sqrt{5}, 6 + \sqrt{5}, 8 - 3i$

Write each function as (a) the product of linear and irreducible quadratic factors and (b) the product of linear factors. Then (c) list all of its zeros. (Example 7)

42. $g(x) = x^4 - 3x^3 - 12x^2 + 20x + 48$

㊸ $g(x) = x^4 - 3x^3 - 12x^2 + 8$

44. $h(x) = x^4 + 2x^3 - 15x^2 + 18x - 216$
45. $f(x) = 4x^4 - 35x^3 + 140x^2 - 295x + 156$
46. $f(x) = 4x^4 - 15x^3 + 43x^2 + 577x + 615$
47. $h(x) = x^4 - 2x^3 - 17x^2 + 4x + 30$
48. $g(x) = x^4 + 31x^2 - 180$

Use the given zero to find all complex zeros of each function. Then write the linear factorization of the function. (Example 8)

49. $h(x) = 2x^5 + x^4 - 7x^3 + 21x^2 - 225x + 108; 3i$
50. $h(x) = 3x^5 - 5x^4 - 13x^3 - 65x^2 - 2200x + 1500; -5i$
51. $g(x) = x^5 - 2x^4 - 13x^3 + 28x^2 + 46x - 60; 3 - i$
52. $g(x) = 4x^5 - 57x^4 + 287x^3 - 547x^2 + 83x + 510; 4 + i$
53. $f(x) = x^5 - 3x^4 - 4x^3 + 12x^2 - 32x + 96; -2i$
54. $g(x) = x^4 - 10x^3 + 35x^2 - 46x + 10; 3 + i$

55. ARCHITECTURE An architect is constructing a scale model of a building that is in the shape of a pyramid.
    - **a.** If the height of the scale model is 9 inches less than its length and its base is a square, write a polynomial function that describes the volume of the model in terms of its length.
    - **b.** If the volume of the model is 6300 cubic inches, write an equation describing the situation.
    - **c.** What are the dimensions of the scale model?

56. CONSTRUCTION The height of a tunnel that is under construction is 1 foot more than half its width and its length is 32 feet more than 324 times its width. If the volume of the tunnel is 62,231,040 cubic feet and it is a rectangular prism, find the length, width, and height.

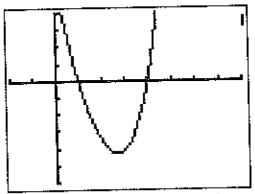
Write a polynomial function of least degree with integer coefficients that has the given number as a zero.

57. $\sqrt[3]{6}$ **58.** $\sqrt[3]{\quad}\ \ 5$

59. $-\sqrt[3]{2}$ **60.** $-\sqrt[3]{7}$

Use each graph to write g as the product of linear factors. Then list all of its zeros.

61. $g(x) = 3x^4 - 15x^3 + 87x^2 - 375x + 300$



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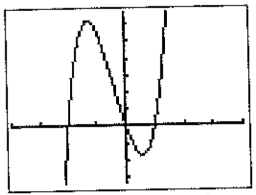
62. $g(x) = 2x^5 + 2x^4 + 28x^3 + 32x^2 - 64x$



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Determine all rational zeros of the function.

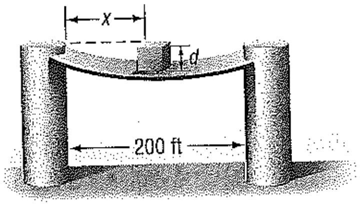
63. $h(x) = 6x^3 - 6x^2 + 12$

64. $f(y) = \frac{1}{4}y^4 + \frac{1}{2}y^3 - y^2 + 2y - 8$

65. $w(z) = z^4 - 10z^3 + 30z^2 - 10z + 29$

66. $b(a) = a^5 - \frac{5}{6}a^4 + \frac{2}{3}a^3 - \frac{2}{3}a^2 - \frac{1}{3}a + \frac{1}{6}$

(67) ENGINEERING A steel beam is supported by two pilings 200 feet apart. If a weight is placed x feet from the piling on the left, a vertical deflection represented by $d = 0.0000008x^2(200 - x)$ occurs. How far is the weight from the piling if the vertical deflection is 0.8 feet?



Write each polynomial as the product of linear and irreducible quadratic factors.

68. $x^3 - 3$ **69.** $x^3 + 16$

70. $8x^3 + 9$ **71.** $27x^6 + 4$

72. MULTIPLE REPRESENTATIONS In this problem, you will explore even- and odd-degree polynomial functions.

- **a. ANALYTICAL** Identify the degree and number of zeros of each polynomial function.
  - **i.** $f(x) = x^3 - x^2 + 9x - 9$
  - **ii.** $g(x) = 2x^5 + x^4 - 32x - 16$
  - **iii.** $h(x) = 5x^3 + 2x^2 - 13x + 6$
  - **iv.** $f(x) = x^4 + 25x^2 + 144$
  - **v.** $h(x) = 3x^6 + 5x^5 + 46x^4 + 80x^3 - 32x^2$
  - **vi.** $g(x) = 4x^4 - 11x^3 + 10x^2 - 11x + 6$
- **b. NUMERICAL** Find the zeros of each function.
- **c. VERBAL** Does an odd-degree function have to have a minimum number of real zeros? Explain.

H.O.T. Problems Use Higher-Order Thinking Skills

73. ERROR ANALYSIS Angie and Julius are using the Rational Zeros Theorem to find all the possible rational zeros of $f(x) = 7x^2 + 2x^3 - 5x - 3$. Angie thinks the possible zeros are $\pm\frac{1}{7}, \pm\frac{3}{7}, \pm1, \pm3$. Julius thinks they are $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm1, \pm3$. Is either of them correct? Explain your reasoning.

74. REASONING Explain why $g(x) = x^9 - x^8 + x^5 + x^3 - x^2 + 2$ must have a root between $x = -1$ and $x = 0$.

75. CHALLENGE Use $f(x) = x^2 + x - 6, f(x) = x^3 + 8x^2 + 19x + 12$, and $f(x) = x^4 - 2x^3 - 21x^2 + 22x + 40$ to make a conjecture about the relationship between the graphs and zeros of $f(x)$ and the graphs and zeros of each of the following.

a. $-f(x)$ **b.** $f(-x)$

76. OPEN ENDED Write a function of 4th degree with an imaginary zero and an irrational zero.

77. REASONING Determine whether the statement is *true* or *false*. If false, provide a counterexample.
A third-degree polynomial with real coefficients has at least one nonreal zero.

CHALLENGE **Find the zeros of each function if $h(x)$ has zeros at x_1, x_2, and x_3.**

78. $c(x) = 7h(x)$ **79.** $k(x) = h(3x)$

80. $g(x) = h(x - 2)$ **81.** $f(x) = h(-x)$

82. REASONING If $x - c$ is a factor of $f(x) = a_1x^5 - a_2x^4 + \ldots$, what value must c be greater than or equal to in order to be an upper bound for the zeros of $f(x)$? Assume a $\neq 0$. Explain your reasoning.

83. WRITING IN MATH Explain why a polynomial with real coefficients and one imaginary zero must have at least two imaginary zeros.

Spiral Review

Divide using synthetic division. (Lesson 2-3)

84. $(x^3 - 9x^2 + 27x - 28) \div (x - 3)$

85. $(x^4 + x^3 - 1) \div (x - 2)$

86. $(3x^4 - 2x^3 + 5x^2 - 4x - 2) \div (x + 1)$

87. $(2x^3 - 2x - 3) \div (x - 1)$

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test. (Lesson 2-2)

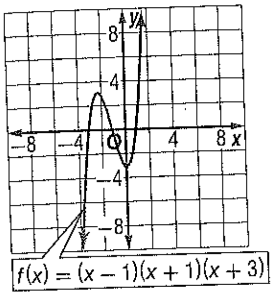
88. $f(x) = -4x^7 + 3x^4 + 6$

89. $f(x) = 4x^6 + 2x^5 + 7x^2$

90. $g(x) = 3x^4 + 5x^5 - 11$

Estimate to the nearest 0.5 unit and classify the extrema for the graph of each function. Support the answers numerically. (Lesson 1-4)

91.



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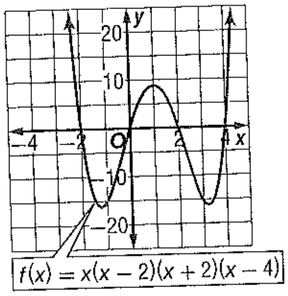
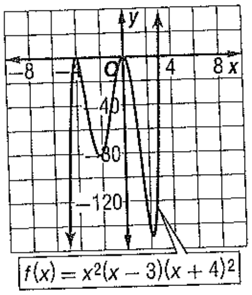


93.



94. FINANCE Investors choose different stocks to comprise a balanced portfolio. The matrices show the prices of one share of each of several stocks on the first business day of July, August, and September. (Lesson 0-6)

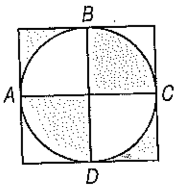
$$\begin{array}{cccc} & \text{July} & \text{August} & \text{September} \\ \text{Stock A} & [33.81 & 30.94 & 27.25] \\ \text{Stock B} & [15.06 & 13.25 & 8.75] \\ \text{Stock C} & [54 & 54 & 46.44] \\ \text{Stock D} & [52.06 & 44.69 & 34.38] \end{array}$$

a. Mrs. Rivera owns 42 shares of stock A, 59 shares of stock B, 21 shares of stock C, and 18 shares of stock D. Write a row matrix to represent Mrs. Rivera's portfolio.

b. Use matrix multiplication to find the total value of Mrs. Rivera's portfolio for each month to the nearest cent.

Skills Review for Standardized Tests

95. SAT/ACT A circle is inscribed in a square and intersects the square at points A, B, C, and D. If $AC = 12$, what is the total area of the shaded regions?



A 18

B 36

C 18π

D 24π

E 72

96. REVIEW $f(x) = x^2 - 4x + 3$ has a relative minimum located at which of the following x-values?

F −2

G 2

H 3

J 4

97. Find all of the zeros of $p(x) = x^3 + 2x^2 - 3x + 20$.

A $-4, 1 + 2i, 1 - 2i$

B $1, 4 + i, 4 - i$

C $-1, 1, 4 + i, 4 - i$

D $4, 1 + i, 1 - i$

98. REVIEW Which expression is equivalent to $(t^2 + 3t - 9)(5 - t)^{-1}$?

F $t + 8 - \frac{31}{5 - t}$

G $-t - 8$

H $-t - 8 + \frac{31}{5 - t}$

J $-t - 8 - \frac{31}{5 - t}$

LESSON 2-5 Rational Functions

:: Then

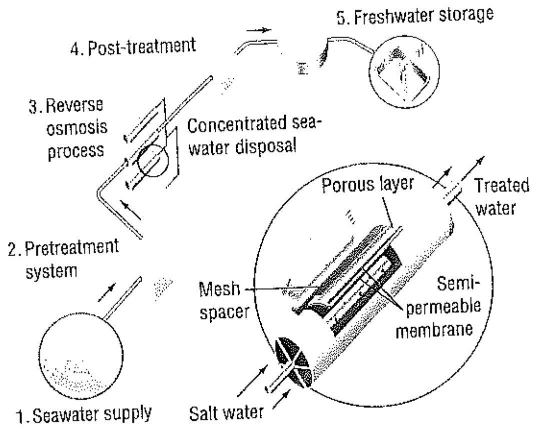
- You identified points of discontinuity and end behavior of graphs of functions using limits. (Lesson 1-3)

:: Now

- 1 Analyze and graph rational functions.
- 2 Solve rational equations.

:: Why?

- Water desalination, or removing the salt from sea water, is currently in use in areas of the world with limited water availability and on many ships and submarines. It is also being considered as an alternative for providing water in the future. The cost for various extents of desalination can be modeled using rational functions.



New Vocabulary

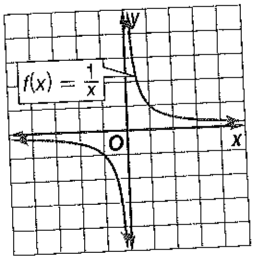
rational function
asymptote
vertical asymptote
horizontal asymptote
oblique asymptote
holes

1 Rational Functions

A **rational function** $f(x)$ is the quotient of two polynomial functions $a(x)$ and $b(x)$, where b is nonzero.

$$f(x) = \frac{a(x)}{b(x)},\ b(x) \neq 0$$

The domain of a rational function is the set of all real numbers excluding those values for which $b(x) = 0$ or the zeros of $b(x)$.



One of the simplest rational functions is the reciprocal function $f(x) = \frac{1}{x}$. The graph of the reciprocal function, like many rational functions, has branches that approach specific x- and y-values. The lines representing these values are called **asymptotes.**

The reciprocal function is undefined when $x = 0$, so its domain is $(-\infty, 0)$ or $(0, \infty)$. The behavior of $f(x) = \frac{1}{x}$ to the left (0^-) and right (0^+) of $x = 0$ can be described using limits.

$$\lim_{x \to 0^-} f(x) = -\infty \qquad \lim_{x \to 0^+} f(x) = \infty$$

From Lesson 1-3, you should recognize 0 as a point of infinite discontinuity in the domain of f. The line $x = 0$ in Figure 2.5.1 is called a *vertical asymptote* of the graph of f. The end behavior of f can be also be described using limits.

$$\lim_{x \to -\infty} f(x) = 0 \qquad \lim_{x \to +\infty} f(x) = 0$$

The line $y = 0$ in Figure 2.5.2 is called a *horizontal asymptote* of the graph of f.

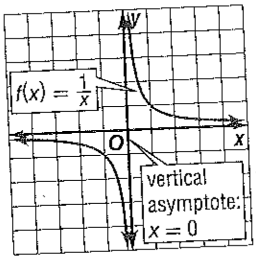


Figure 2.5.1

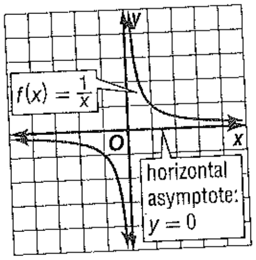


Figure 2.5.2

These definitions of vertical and horizontal asymptotes can be generalized.

You can use your knowledge of limits, discontinuity, and end behavior to determine the vertical and horizontal asymptotes, if any, of a rational function.

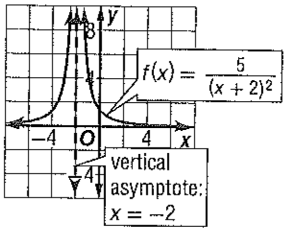
ReadingMath

Limit Notation The expression $\lim_{x\to c^-} f(x)$ is read as the *limit of f of x as x approaches c from the left* and the expression $\lim_{x\to c^+} f(x)$ is read as the *limit of f of x as x approaches c from the right.*

KeyConcept Vertical and Horizontal Asymptotes

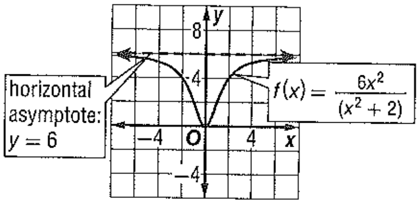
Words The line $x = c$ is a vertical asymptote of the graph of f if $\lim_{x\to c^-} f(x) = \pm\infty$ or $\lim_{x\to c^+} f(x) = \pm\infty$.

Example



Words The line $y = c$ is a horizontal asymptote of the graph of f if $\lim_{x\to -\infty} f(x) = c$ or $\lim_{x\to \infty} f(x) = c$.

Example



Example 1 Find Vertical and Horizontal Asymptotes

Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any.

a. $f(x) = \frac{x+4}{x-3}$

Step 1 **Find the domain.**

The function is undefined at the real zero of the denominator $b(x) = x - 3$. The real zero of $b(x)$ is 3. Therefore, the domain of f is all real numbers except $x = 3$.

Step 2 **Find the asymptotes, if any.**

Check for vertical asymptotes.

Determine whether $x = 3$ is a point of infinite discontinuity. Find the limit as x approaches 3 from the left and the right.

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	−69	−699	−6999	undefined	7001	701	71

Because $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$, you know that $x = 3$ is a vertical asymptote of f.

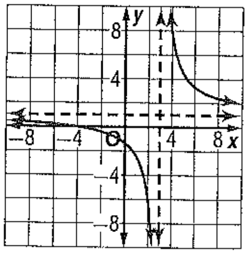
Check for horizontal asymptotes.

Use a table to examine the end behavior of $f(x)$.

x	−10,000	−1000	−100	0	100	1000	10,000
$f(x)$	0.9993	0.9930	0.9320	−1.33	1.0722	1.0070	1.0007

The table suggests that $\lim_{x\to -\infty} f(x) = \lim_{x\to \infty} f(x) = 1$. Therefore, you know that $y = 1$ is a horizontal asymptote of f.

CHECK The graph of $f(x) = \frac{x+4}{x-3}$ shown supports each of these findings. ✓



b. $g(x) = \dfrac{8x^2 + 5}{4x^2 + 1}$

Step 1 The zeros of the denominator $b(x) = 4x^2 + 1$ are imaginary, so the domain of g is all real numbers.

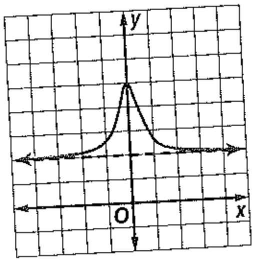
Step 2 Because the domain of g is all real numbers, the function has no vertical asymptotes. Using division, you can determine that

$$g(x) = \frac{8x^2 + 5}{4x^2 + 1} = 2 + \frac{3}{4x^2 + 1}.$$

As the value of $|x|$ increases, $4x^2 + 1$ becomes an increasing large positive number and $\dfrac{3}{4x^2 + 1}$ decreases, approaching 0. Therefore,

$$\lim_{x \to -\infty} g(x) = \lim_{x \to \infty} g(x) = 2 + 0 \text{ or } 2.$$

CHECK You can use a table of values to support this reasoning. The graph of $g(x) = \dfrac{8x^2 + 5}{4x^2 + 1}$ shown also supports each of these findings. ✓



GuidedPractice

Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any.

1A. $m(x) = \dfrac{15x + 3}{x + 5}$

1B. $h(x) = \dfrac{x^2 - x - 6}{x + 4}$

The analysis in Example 1 suggests a connection between the end behavior of a function and its horizontal asymptote. This relationship, along with other features of the graphs of rational functions, is summarized below.

KeyConcept Graphs of Rational Functions

If f is the rational function given by

$$f(x) = \frac{a(x)}{b(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

where $b(x) \neq 0$ and $a(x)$ and $b(x)$ have no common factors other than ± 1, then the graph of f has the following characteristics.

Vertical Asymptotes Vertical asymptotes may occur at the real zeros of $b(x)$.

Horizontal Asymptote The graph has either one or no horizontal asymptotes as determined by comparing the degree n of $a(x)$ to the degree m of $b(x)$.

- If $n < m$, the horizontal asymptote is $y = 0$.
- If $n = m$, the horizontal asymptote is $y = \dfrac{a_n}{b_m}$.
- If $n > m$, there is no horizontal asymptote.

Intercepts The x-intercepts, if any, occur at the real zeros of $a(x)$. The y-intercept, if it exists, is the value of f when $x = 0$.

StudyTip

Poles A vertical asymptote in the graph of a rational function is also called a *pole* of the function.

StudyTip

Test Intervals A rational function can change sign at its zeros and its undefined values, so when these x-values are ordered, they divide the domain of the function into test intervals that can help you determine if the graph lies above or below the x-axis.

To graph a rational function, simplify f, if possible, and then follow these steps.

Step 1 Find the domain.

Step 2 Find and sketch the asymptotes, if any.

Step 3 Find and plot the x-intercepts and y-intercept, if any.

Step 4 Find and plot at least one point in the *test intervals* determined by any x-intercepts and vertical asymptotes.

Example 2 Graph Rational Functions: $n < m$ and $n > m$

For each function, determine any vertical and horizontal asymptotes and intercepts. Then graph the function, and state its domain.

a. $g(x) = \frac{6}{x+3}$

Step 1 The function is undefined at $b(x) = 0$, so the domain is $\{x \mid x \neq -3, x \in \mathbb{R}\}$.

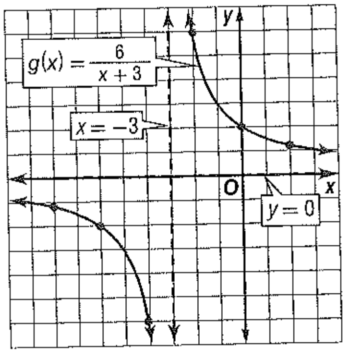
Step 2 There is a vertical asymptote at $x = -3$.

The degree of the polynomial in the numerator is 0, and the degree of the polynomial in the denominator is 1. Because $0 < 1$, the graph of g has a horizontal asymptote at $y = 0$.

Step 3 The polynomial in the numerator has no real zeros, so g has no x-intercepts. Because $g(0) = 2$, the y-intercept is 2.

Step 4 Graph the asymptotes and intercepts. Then choose x-values that fall in the test intervals determined by the vertical asymptote to find additional points to plot on the graph. Use smooth curves to complete the graph.

Interval	x	$(x, g(x))$
$(-\infty, -3)$	−8	(−8, −1.2)
	−6	(−6, −2)
	−4	(−4, −6)
$(-3, \infty)$	−2	(−2, 6)
	2	(2, 1.2)



StudyTip

Hyperbola The graphs of the reciprocal functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{6}{x+3}$ are called *hyperbolas*. You will learn more about hyperbolas in Chapter 7.

b. $k(x) = \frac{x^2 - 7x + 10}{x - 3}$

Factoring the numerator yields $k(x) = \frac{(x-2)(x-5)}{x-3}$. Notice that the numerator and denominator have no common factors, so the expression is in simplest form.

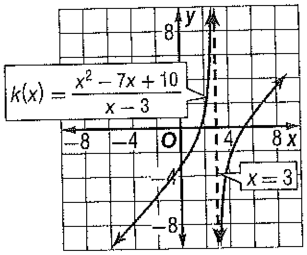
Step 1 The function is undefined at $b(x) = 0$, so the domain is $\{x \mid x \neq 3, x \in \mathbb{R}\}$.

Step 2 There is a vertical asymptote at $x = 3$.

Compare the degrees of the numerator and denominator. Because $2 > 1$, there is no horizontal asymptote.

Step 3 The numerator has zeros at $x = 2$ and $x = 5$, so the x-intercepts are 2 and 5. $k(0) = -\frac{10}{3}$, so the y-intercept is at about −3.3.

Step 4 Graph the asymptotes and intercepts. Then find and plot points in the test intervals determined by the intercepts and vertical asymptotes: $(-\infty, 0)$, $(0, 3)$, $(3, \infty)$. Use smooth curves to complete the graph.



GuidedPractice

2A. $h(x) = \frac{2}{x^2 + 2x - 3}$

2B. $n(x) = \frac{x}{x^2 + x - 2}$

In Example 3, the degree of the numerator is *equal to* the degree of the denominator.

Example 3 Graph a Rational Function: $n = m$

Determine any vertical and horizontal asymptotes and intercepts for $f(x) = \frac{3x^2 - 3}{x^2 - 9}$. Then graph the function, and state its domain.

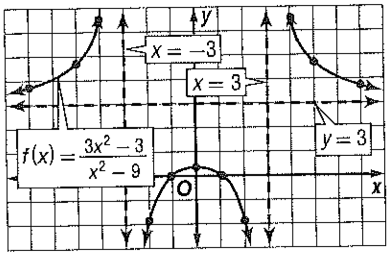
Factoring both numerator and denominator yields $f(x) = \frac{3(x-1)(x+1)}{(x-3)(x+3)}$ with no common factors.

Step 1 The function is undefined at $b(x) = 0$, so the domain is $\{x \mid x \neq -3, 3, x \in \mathbb{R}\}$.

Step 2 There are vertical asymptotes at $x = 3$ and $x = -3$.
There is a horizontal asymptote at $y = \frac{3}{1}$ or $y = 3$, the ratio of the leading coefficients of the numerator and denominator, because the degrees of the polynomials are equal.

Step 3 The x-intercepts are 1 and -1, the zeros of the numerator. The y-intercept is $\frac{1}{3}$ because $f(0) = \frac{1}{3}$.

Step 4 Graph the asymptotes and intercepts. Then find and plot points in the test intervals $(-\infty, -3)$, $(-3, -1)$, $(-1, 1)$, $(1, 3)$, and $(3, \infty)$.



GuidedPractice

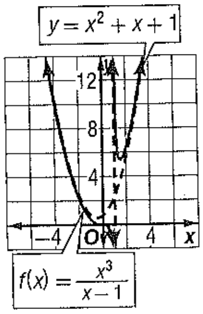
For each function, determine any vertical and horizontal asymptotes and intercepts. Then graph the function and state its domain.

3A. $h(x) = \frac{x - 6}{x + 2}$

3B. $h(x) = \frac{x^2 - 4}{5x^2 - 5}$

StudyTip

Nonlinear Asymptotes
Horizontal, vertical, and oblique asymptotes are all linear. A rational function can also have a nonlinear asymptote. For example, the graph of $f(x) = \frac{x^3}{x - 1}$ has a quadratic asymptote.



When the degree of the numerator is *exactly one more* than the degree of the denominator, the graph has a *slant* or **oblique asymptote**.

KeyConcept Oblique Asymptotes

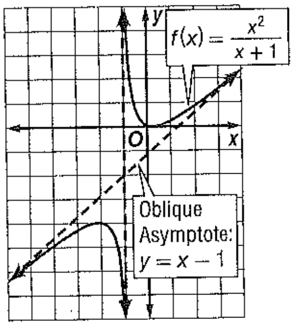
If f is the rational function given by

$$f(x) = \frac{a(x)}{b(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0},$$

where $b(x)$ has a degree greater than 0 and $a(x)$ and $b(x)$ have no common factors other than 1, then the graph of f has an oblique asymptote if $n = m + 1$. The function for the oblique asymptote is the quotient polynomial $q(x)$ resulting from the division of $a(x)$ by $b(x)$.

$$f(x) = \frac{a(x)}{b(x)} = \underbrace{q(x)}_{\text{function for oblique asymptote}} + \frac{r(x)}{b(x)}$$

Example



Example 4 Graph a Rational Function: $n = m + 1$

Determine any asymptotes and intercepts for $f(x) = \frac{2x^3}{x^2 + x - 12}$. Then graph the function, and state its domain.

Factoring the denominator yields $f(x) = \frac{2x^3}{(x + 4)(x - 3)}$.

Step 1 The function is undefined at $b(x) = 0$, so the domain is $\{x \mid x \neq -4, 3, x \in \mathbb{R}\}$.

Step 2 There are vertical asymptotes at $x = -4$ and $x = 3$.

The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote.

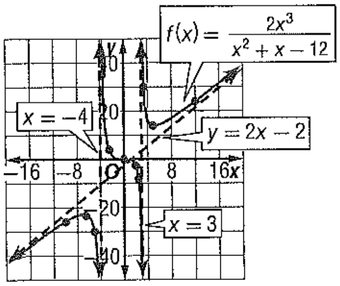
Because the degree of the numerator is exactly one more than the degree of the denominator, *f* has a slant asymptote. Using polynomial division, you can write the following.

$$f(x) = \frac{2x^3}{x^2 + x - 12}$$
$$= 2x - 2 + \frac{26x - 24}{x^2 + x - 12}$$

Therefore, the equation of the slant asymptote is $y = 2x - 2$.

Step 3 The *x*- and *y*-intercepts are 0 because 0 is the zero of the numerator and $f(0) = 0$.

Step 4 Graph the asymptotes and intercepts. Then find and plot points in the test intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 3)$, and $(3, \infty)$.



StudyTip

End Behavior Asymptote In Example 4, the graph of *f approaches* the slant asymptote $y = 2x - 2$ as $x \to \pm\infty$. Between the vertical asymptotes $x = -4$ and $x = 3$, however, the graph *crosses* the line $y = 2x - 2$. For this reason, a slant or horizontal asymptote is sometimes referred to as an *end behavior asymptote.*

GuidedPractice

For each function, determine any asymptotes and intercepts. Then graph the function and state its domain.

4A. $h(x) = \frac{x^2 + 3x - 3}{x + 4}$

4B. $p(x) = \frac{x^2 - 4x + 1}{2x - 3}$

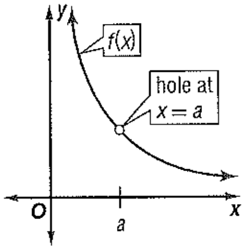
When the numerator and denominator of a rational function have common factors, the graph of the function has removable discontinuities called **holes**, at the zeros of the common factors. Be sure to indicate these points of discontinuity when you graph the function.

StudyTip

Removable and Nonremovable Discontinuities If the function is not continuous at $x = a$, but could be made continuous at that point by simplifying, then the function has a *removable discontinuity* at $x = a$. Otherwise, it has a *nonremovable discontinuity* $x = a$.

$$f(x) = \frac{\cancel{(x - a)}(x - b)}{\cancel{(x - a)}(x - c)}$$

Divide out the common factor in the numerator and denominator. The zero of $x - a$ is a.



Example 5 Graph a Rational Function with Common Factors

Determine any vertical and horizontal asymptotes, holes, and intercepts for $h(x) = \frac{x^2 - 4}{x^2 - 2x - 8}$. Then graph the function, and state its domain.

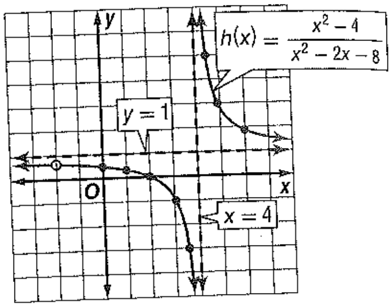
Factoring both the numerator and denominator yields $h(x) = \frac{(x-2)\cancel{(x+2)}}{(x-4)\cancel{(x+2)}}$ or $\frac{x-2}{x-4}$, $x \neq -2$.

Step 1 The function is undefined at $b(x) = 0$, so the domain is $\{x \mid x \neq -2, 4, x \in \mathbb{R}\}$.

Step 2 There is a vertical asymptote at $x = 4$, the real zero of the simplified denominator. There is a horizontal asymptote at $y = \frac{1}{1}$ or 1, the ratio of the leading coefficients of the numerator and denominator, because the degrees of the polynomials are equal.

Step 3 The x-intercept is 2, the zero of the simplified numerator. The y-intercept is $\frac{1}{2}$ because $h(0) = \frac{1}{2}$.

Step 4 Graph the asymptotes and intercepts. Then find and plot points in the test intervals $(-\infty, 2)$, $(2, 4)$, and $(4, \infty)$. There is a hole at $\left(-2, \frac{2}{3}\right)$ because the original function is undefined when $x = -2$.



StudyTip

Hole For Example 5, $x + 2$ was divided out of the original expression. Substitute -2 into the new expression.

$$h(-2) = \frac{(-2) - 2}{(-2) - 4}$$
$$= \frac{-4}{-6} \text{ or } \frac{2}{3}$$

There is a hole at $\left(-2, \frac{2}{3}\right)$.

GuidedPractice

For each function, determine any vertical and horizontal asymptotes, holes, and intercepts. Then graph the function and state its domain.

5A. $g(x) = \frac{x^2 + 10x + 24}{x^2 + x - 12}$

5B. $c(x) = \frac{x^2 - 2x - 3}{x^2 - 4x - 5}$

2 Rational Equations

Rational equations involving fractions can be solved by multiplying each term in the equation by the least common denominator (LCD) of all the terms of the equation.

Example 6 Solve a Rational Equation

Solve $x + \frac{6}{x-8} = 0$.

$x + \frac{6}{x-8} = 0$ Original equation

$x(x-8) + \frac{6}{x-8}(x-8) = 0(x-8)$ Multiply by the LCD, $x - 8$

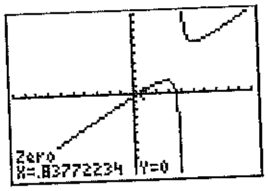
$x^2 - 8x + 6 = 0$ Distributive Property

$x = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(6)}}{2(1)}$ Quadratic Formula

$x = \frac{8 \pm 2\sqrt{10}}{2}$ or $4 \pm \sqrt{10}$ Simplify.

StudyTip

Check for Reasonableness You can also check the result in Example 6 by using a graphing calculator to graph $y = x + \frac{6}{x-8}$. Use the CALC menu to locate the zeros. Because the zeros of the graph appear to be at about $x = 7.16$ and $x = 0.84$, the solution is reasonable.



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GuidedPractice

Solve each equation.

6A. $\frac{20}{x+3} - 4 = 0$

6B. $\frac{9x}{x-2} = 6$

StudyTip

Intersection You can use the intersection feature of your graphing calculator to solve a rational equation by graphing each side of the equation and finding all of the intersections of the two graphs.

Solving a rational equation can produce extraneous solutions. Always check your answers in the original equation.

Example 7 Solve a Rational Equation with Extraneous Solutions

Solve $\frac{4}{x^2 - 6x + 8} = \frac{3x}{x-2} + \frac{2}{x-4}$.

The LCD of the expressions is $(x-2)(x-4)$, which are the factors of $x^2 - 6x + 8$.

$$\frac{4}{x^2 - 6x + 8} = \frac{3x}{x-2} + \frac{2}{x-4} \qquad \text{Original equation}$$

$$(x-2)(x-4)\,\frac{4}{x^2 - 6x + 8} = (x-2)(x-4)\left(\frac{3x}{x-2} + \frac{2}{x-4}\right) \qquad \text{Multiply by the LCD.}$$

$$4 = 3x(x-4) + 2(x-2) \qquad \text{Distributive Property}$$

$$4 = 3x^2 - 10x - 4 \qquad \text{Distributive Property}$$

$$0 = 3x^2 - 10x - 8 \qquad \text{Subtract 4 from each side.}$$

$$0 = (3x+2)(x-4) \qquad \text{Factor.}$$

$$x = -\frac{2}{3} \text{ or } x = 4 \qquad \text{Solve.}$$

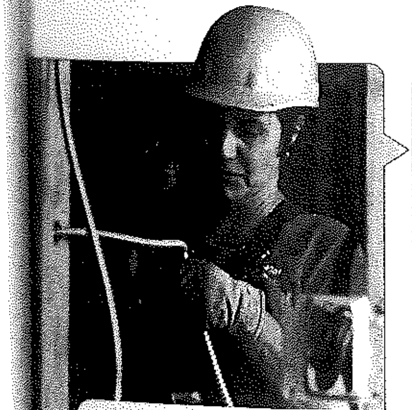
Because the original equation is not defined when $x = 4$, you can eliminate this extraneous solution. So, the only solution is $-\frac{2}{3}$.

GuidedPractice

Solve each equation.

7A. $\frac{2x}{x+3} + \frac{3}{x-6} = \frac{27}{x^2 - 3x - 18}$

7B. $-\frac{12}{x^2 + 6x} = \frac{2}{x+6} + \frac{x-2}{x}$

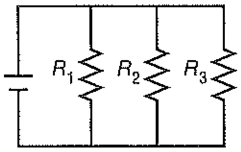


Real-World Career

Electrician Electricians install and maintain various components of electricity, such as wiring and fuses. They must maintain compliance with national, state, and local codes. Most electricians complete an apprenticeship program that includes both classroom instruction and on-the-job training.

Real-World Example 8 Solve a Rational Equation

ELECTRICITY **The diagram of an electric circuit shows three parallel resistors. If R is the equivalent resistance of the three resistors, then $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. In this circuit, R_1 is twice the resistance of R_2, and R_3 equals 20 ohms. Suppose the equivalent resistance is equal to 10 ohms. Find R_1 and R_2.**



$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \qquad \text{Original equation}$$

$$\frac{1}{10} = \frac{1}{2R_2} + \frac{1}{R_2} + \frac{1}{20} \qquad R = 10,\ R_1 = 2R_2, \text{ and } R_3 = 20$$

$$\frac{1}{20} = \frac{1}{2R_2} + \frac{1}{R_2} \qquad \text{Subtract } \frac{1}{20} \text{ from each side.}$$

$$(20R_2)\,\frac{1}{20} = (20R_2)\left(\frac{1}{2R_2} + \frac{1}{R_2}\right) \qquad \text{Multiply each side by the LCD, } 20R_2.$$

$$R_2 = 10 + 20 \text{ or } 30 \qquad \text{Simplify}$$

R_2 is 30 ohms and $R_1 = 2R_2$ or 60 ohms.

GuidedPractice

8. ELECTRONICS Suppose the current I, in amps, in an electric circuit is given by the formula $I = t + \frac{1}{10 - t}$, where t is time in seconds. At what time is the current 1 amp?

Exercises

= Step-by-Step Solutions begin on page R29.

Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any. (Example 1)

1. $f(x) = \frac{x^2 - 2}{x^2 - 4}$

2. $h(x) = \frac{x^3 - 8}{x + 4}$

3. $f(x) = \frac{x(x-1)(x+2)^2}{(x+3)(x-4)}$

4. $g(x) = \frac{x - 6}{(x+3)(x+5)}$

5. $h(x) = \frac{2x^2 - 4x + 1}{x^2 + 2x}$

6. $f(x) = \frac{x^2 + 9x + 20}{x - 4}$

7. $h(x) = \frac{(x-1)(x+1)}{(x-2)^2(x+4)^2}$

8. $g(x) = \frac{(x-4)(x+2)}{(x+1)(x-3)}$

For each function, determine any asymptotes and intercepts. Then graph the function, and state its domain. (Examples 2–5)

9. $f(x) = \frac{(x+2)(x-3)}{(x+4)(x-5)}$

10. $g(x) = \frac{(2x+3)(x-6)}{(x+2)(x-1)}$

11. $f(x) = \frac{8}{(x-2)(x+2)}$

12. $f(x) = \frac{x + 2}{x(x-6)}$

13. $g(x) = \frac{(x+2)(x+5)}{(x+5)^2(x-6)}$

14. $h(x) = \frac{(x+6)(x+4)}{x(x-5)(x+2)}$

15. $h(x) = \frac{x^2(x-2)(x+5)}{x^2 + 4x + 3}$

16. $f(x) = \frac{x(x+6)^2(x-4)}{x^2 - 5x - 24}$

17. $f(x) = \frac{x - 8}{x^2 + 4x + 5}$

18. $g(x) = \frac{-4}{x^2 + 6}$

19. SALES The business plan for a new car wash projects that profits in thousands of dollars will be modeled by the function $p(z) = \frac{3z^2 - 3}{2z^2 + 7z + 5}$, where z is the week of operation and $z = 0$ represents opening. (Example 4)
   a. State the domain of the function.
   b. Determine any vertical and horizontal asymptotes and intercepts for $p(z)$.
   c. Graph the function.

For each function, determine any asymptotes, holes, and intercepts. Then graph the function and state its domain. (Examples 2–5)

20. $h(x) = \frac{3x - 4}{x^3}$

21. $h(x) = \frac{4x^2 - 2x + 1}{3x^3 + 4}$

22. $f(x) = \frac{x^2 + 2x - 15}{x^2 + 4x + 3}$

23. $g(x) = \frac{x + 7}{x - 4}$

24. $h(x) = \frac{x^3}{x + 3}$

25. $g(x) = \frac{x^3 + 3x^2 + 2x}{x - 4}$

26. $f(x) = \frac{x^2 - 4x - 21}{x^3 + 2x^2 - 5x - 6}$

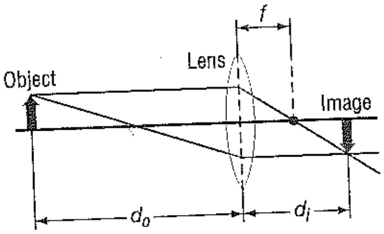
㉗ $g(x) = \frac{x^2 - 4}{x^3 + x^2 - 4x - 4}$

28. $f(x) = \frac{(x+4)(x-1)}{(x-1)(x+3)}$

29. $g(x) = \frac{(2x+1)(x-5)}{(x-5)(x+4)^2}$

30. STATISTICS A number x is said to be the *harmonic mean* of y and z if $\frac{1}{x}$ is the average of $\frac{1}{y}$ and $\frac{1}{z}$. (Example 7)
   a. Write an equation for which the solution is the harmonic mean of 30 and 45.
   b. Find the harmonic mean of 30 and 45.

31. OPTICS The lens equation is $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$, where f is the focal length, d_i is the distance from the lens to the image, and d_o is the distance from the lens to the object. Suppose the object is 32 centimeters from the lens and the focal length is 8 centimeters. (Example 7)



   a. Write a rational equation to model the situation.
   b. Find the distance from the lens to the image.

Solve each equation. (Examples 6–8)

32. $y + \frac{6}{y} = 5$

33. $\frac{8}{z} - z = 4$

34. $\frac{x - 1}{2x - 4} + \frac{x + 2}{3x} = 1$

35. $\frac{2}{y + 2} - \frac{y}{2 - y} = \frac{y^2 + 4}{y^2 - 4}$

36. $\frac{3}{x} + \frac{2}{x + 1} = \frac{23}{x^2 + x}$

37. $\frac{4}{x - 2} - \frac{2}{x} = \frac{14}{x^2 - 2x}$

38. $\frac{x}{x + 1} - \frac{x - 1}{x} = \frac{1}{20}$

39. $\frac{6}{x - 3} - \frac{4}{x + 2} = \frac{12}{x^2 - x - 6}$

40. $\frac{x - 1}{x - 2} + \frac{3x + 6}{2x + 1} = 3$

41. $\frac{2}{a + 3} - \frac{3}{4 - a} = \frac{2a - 2}{a^2 - a - 12}$

42. WATER The cost per day to remove x percent of the salt from seawater at a desalination plant is $c(x) = \frac{994x}{100 - x}$, where $0 \leq x < 100$.
   a. Graph the function using a graphing calculator.
   b. Graph the line $y = 8000$ and find the intersection with the graph of $c(x)$ to determine what percent of salt can be removed for \$8000 per day.
   c. According to the model, is it feasible for the plant to remove 100% of the salt? Explain your reasoning.

Write a rational function for each set of characteristics.

43. x-intercepts at $x = 0$ and $x = 4$, vertical asymptotes at $x = 1$ and $x = 6$, and a horizontal asymptote at $y = 0$

44. x-intercepts at $x = 2$ and $x = -3$, vertical asymptote at $x = 4$, and point discontinuity at $(-5, 0)$

45. **TRAVEL** When distance and time are held constant, the average rates, in miles per hour, during a round trip can be modeled by $r_2 = \frac{30r_1}{r_1 - 30}$, where r_1 represents the average rate during the first leg of the trip and r_2 represents the average rate during the return trip.

a. Find the vertical and horizontal asymptotes of the function, if any. Verify your answer graphically.

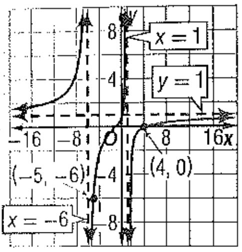
b. Copy and complete the table shown.

r_1	45	50	55	60	65	70
r_2						

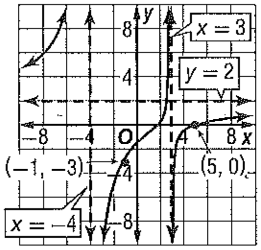
c. Is a domain of $r_1 > 30$ reasonable for this situation? Explain your reasoning.

Use your knowledge of asymptotes and the provided points to express the function represented by each graph.

46.



47.



Use the intersection feature of a graphing calculator to solve each equation.

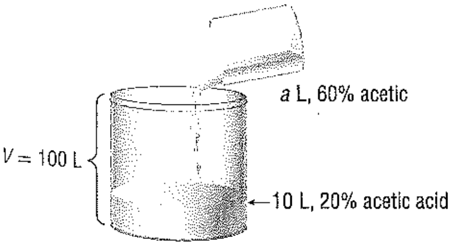
48. $\frac{x^4 - 2x^3 + 1}{x^3 + 6} = 8$

49. $\frac{2x^4 - 5x^2 + 3}{x^4 + 3x^2 - 4} = 1$

50. $\frac{3x^3 - 4x^2 + 8}{4x^4 + 2x - 1} = 2$

51. $\frac{2x^5 - 3x^3 + 5x}{4x^3 + 5x - 12} = 6$

52. **CHEMISTRY** When a 60% acetic acid solution is added to 10 liters of a 20% acetic acid solution in a 100-liter tank, the concentration of the total solution changes.



a. Show that the concentration of the solution is $f(a) = \frac{3a + 10}{5a + 50}$, where a is the volume of the 60% solution.

b. Find the relevant domain of $f(a)$ and the vertical or horizontal asymptotes, if any.

c. Explain the significance of any domain restrictions or asymptotes.

d. Disregarding domain restrictions, are there any additional asymptotes of the function? Explain.

53. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate asymptotes of rational functions.

a. **TABULAR** Copy and complete the table. Determine the horizontal asymptote of each function algebraically.

Function	Horizontal Asymptote
$f(x) = \frac{x^2 - 5x + 4}{x^3 + 2}$	
$h(x) = \frac{x^3 - 3x^2 + 4x - 12}{x^4 - 4}$	
$g(x) = \frac{x^4 - 1}{x^5 + 3}$	

b. **GRAPHICAL** Graph each function and its horizontal asymptote from part a.

c. **TABULAR** Copy and complete the table below. Use the Rational Zero Theorem to help you find the real zeros of the numerator of each function.

Function	Real Zeros of Numerator
$f(x) = \frac{x^2 - 5x + 4}{x^3 + 2}$	
$h(x) = \frac{x^3 - 3x^2 + 4x - 12}{x^4 - 4}$	
$g(x) = \frac{x^4 - 1}{x^5 + 3}$	

d. **VERBAL** Make a conjecture about the behavior of the graph of a rational function when the degree of the denominator is greater than the degree of the numerator and the numerator has at least one real zero.

H.O.T. Problems Use Higher-Order Thinking Skills

54. **REASONING** Given $f(x) = \frac{ax^3 + bx^2 + c}{dx^3 + ex^2 + f}$, will $f(x)$ *sometimes*, *always*, or *never* have a horizontal asymptote at $y = 1$ if a, b, c, d, e, and f are constants with $a \neq 0$ and $d \neq 0$. Explain.

55. **PREWRITE** Design a lesson plan to teach the graphing rational functions topics in this lesson. Make a plan that addresses purpose, audience, a controlling idea, logical sequence, and time frame for completion.

56. **CHALLENGE** Write a rational function that has vertical asymptotes at $x = -2$ and $x = 3$ and an oblique asymptote $y = 3x$.

57. **WRITING IN MATH** Use words, graphs, tables, and equations to show how to graph a rational function.

58. **CHALLENGE** Solve for k so that the rational equation has exactly one extraneous solution and one real solution.

$$\frac{2}{x^2 - 4x + k} = \frac{2x}{x - 1} + \frac{1}{x - 3}$$

59. **WRITING IN MATH** Explain why all of the test intervals must be used in order to get an accurate graph of a rational function.

Spiral Review

List all the possible rational zeros of each function. Then determine which, if any, are zeros. (Lesson 2-4)

60. $f(x) = x^3 + 2x^2 - 5x - 6$ **61.** $f(x) = x^3 - 2x^2 + x + 18$ **62.** $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2$

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$. (Lesson 2-3)

63. $f(x) = x^4 - 2x^3 - 13x^2 + 14x + 24; x - 3, x - 2$

64. $f(x) = 2x^4 - 5x^3 - 11x^2 - 4x; x - 4, 2x - 1$

65. $f(x) = 6x^4 + 59x^3 + 138x^2 - 45x - 50; 3x - 2, x - 5$

66. $f(x) = 4x^4 - 3x^3 - 12x^2 + 17x - 6; 4x - 3; x - 1$

67. $f(x) = 4x^5 + 15x^4 + 12x^3 - 4x^2; x + 2, 4x + 1$

68. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2; x + 1, x - 1$

Graph each function. (Lesson 2-2)

69. $f(x) = (x + 7)^2$ **70.** $f(x) = (x - 4)^3$ **71.** $f(x) = x^4 - 5$

72. RETAIL Sara is shopping at a store that offers \$10 cash back for every \$50 spent. Let $f(x) = \left[\!\left[\frac{x}{50}\right]\!\right]$ and $h(x) = 10x$, where x is the amount of money Sara spends. (Lesson 1-6)

a. If Sara spends money at the store, is the cash back bonus represented by $f[h(x)]$ or $h[f(x)]$? Explain your reasoning.

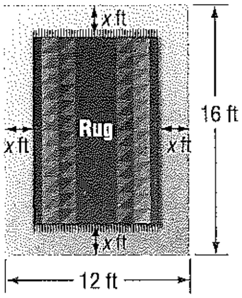
b. Determine the cash back bonus if Sara spends \$312.68 at the store.

73. INTERIOR DESIGN Adrienne Herr is an interior designer. She has been asked to locate an oriental rug for a new corporate office. The rug should cover half of the total floor area with a uniform width surrounding the rug. (Lesson 0-3)

a. If the dimensions of the room are 12 feet by 16 feet, write an equation to model the area of the rug in terms of x.

b. Graph the related function.

c. What are the dimensions of the rug?



Simplify. (Lesson 0-2)

74. $i^{10} + i^2$ **75.** $(2 + 3i) + (-6 + i)$ **76.** $(2.3 + 4.1i) - (-1.2 - 6.3i)$

Skills Review for Standardized Tests

77. SAT/ACT A company sells ground coffee in two sizes of cylindrical containers. The smaller container holds 10 ounces of coffee. If the larger container has twice the radius of the smaller container and 1.5 times the height, how many ounces of coffee does the larger container hold? (The volume of a cylinder is given by the formula $V = \pi r^2 h$.)

A 30 C 60 E 90

B 45 D 75

78. What are the solutions of $1 = \frac{2}{x^2} + \frac{2}{x}$?

F $x = 1, x = -2$ H $x = 1 + \sqrt{3}, x = 1 - \sqrt{3}$

G $x = -2, x = 1$ J $x = \frac{1 + \sqrt{3}}{2}, x = \frac{1 - \sqrt{3}}{2}$

79. REVIEW Alex wanted to determine the average of his 6 test scores. He added the scores correctly to get T but divided by 7 instead of 6. The result was 12 less than his actual average. Which equation could be used to determine the value of T?

A $6T + 12 = 7T$ C $\frac{T}{7} + 12 = \frac{T}{6}$

B $\frac{T}{7} = \frac{T - 12}{6}$ D $\frac{T}{6} = \frac{T - 12}{7}$

80. Diana can put a puzzle together in three hours. Ella can put the same puzzle together in five hours. How long will it take them if they work together?

F $1\frac{3}{8}$ hours H $1\frac{3}{4}$ hours

G $1\frac{5}{8}$ hours J $1\frac{7}{8}$ hours